# Chapter 3
# The Master Theorem

## 3.1 Introduction and Commentary

One of MacMahon's most surprising and valuable contributions to combinatorics was his discovery of the following.

*Master Theorem.* The coefficient of $X_1^{p_1} X_2^{p_2} \ldots X_n^{p_n}$ in the product of linear forms

$$(A_{\binom{1}{1}} X_1 + \ldots + A_{\binom{1}{n}} X_n)^{p_1} (A_{\binom{2}{1}} X_1 + \ldots + A_{\binom{2}{n}} X_n)^{p_2}$$
$$\ldots (A_{\binom{n}{1}} X_1 + \ldots + A_{\binom{n}{n}} X_n)^{p_n}$$

is identical with the coefficient of $X_1^{p_1} X_2^{p_2} \ldots X_n^{p_n}$ in

$$\begin{vmatrix} 1 - A_{\binom{1}{1}} X_1 & - A_{\binom{1}{2}} X_2 \ldots & - A_{\binom{1}{n}} X_n \\ - A_{\binom{2}{1}} X_1 & 1 - A_{\binom{2}{2}} X_2 \ldots & - A_{\binom{2}{n}} X_n \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ - A_{\binom{n}{1}} X_1 & - A_{\binom{n}{2}} X_2 \ldots & 1 - A_{\binom{n}{n}} X_n \end{vmatrix}^{-1}$$

$$= (\det(\delta_{ij} - A_{\binom{i}{j}} X_j))^{-1},$$

where $A_{\binom{i}{j}}$ is our special notation for the doubly subscripted coefficient $A_{ij}$.

This theorem is studied in great detail in [45], and MacMahon's motivation to do so may be clearly seen in his first memoir on compositions [42]. In fact, we find the most extensive examples of the power and usefulness of the Master Theorem in these two papers, where MacMahon applies his result to numerous permutation problems.

A number of proofs of the Master Theorem have been given since MacMahon's original proof. Probably the most important contribution was made by D. Foata (1965) who explained, in his own proof, the real combinatorial significance of the Master Theorem. Subsequently, Foata (in

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The material in this chapter corresponds to section III, chapters II and IV in *Combinatory Analysis*.

collaboration with P. Cartier) presented a simplified approach to his proof (Cartier and Foata, 1969). Cartier (1972) provided three striking proofs, one using symmetric functions, one using the symmetric group, and one using homological algebra (Lefschetz's Lemma). I. J. Good (1962a) and J. Percus (1971) each provided a proof using multiple integral complex variable techniques.

While the Master Theorem is historically the first really important theorem of this type, other mathematicians (back to Jacobi) have studied related problems. An excellent account of such researches was given by T. Muir (1903), who placed the Master Theorem in a setting which also anticipates recent work of P. Whittle (1956).

Applications of the Master Theorem have been discussed extensively by Percus (1971), and also by L. Carlitz (1974, 1977), D. Foata (1965), P. Cartier and D. Foata (1969), I. J. Good (1962b), M. Hall (1958), and H. S. Wilf (1968). Askey, Ismail, and Rashed (1975) and Askey and Ismail (1976) have presented a particularly interesting relationship between the Master Theorem and positivity problems in analysis.

To give some of the flavor of recent work on the Master Theorem, we shall present a modified account of the proof of Foata (1965). (See also Cartier and Foata, 1969) in section 4.2. In section 4.3 we shall discuss a conjecture of F. J. Dyson (1962) and one of the three subsequent proofs (I. J. Good, 1970); D. Foata has remarked that Dyson's conjecture is closely related to the Master Theorem and has suggested that many further results of combinatorial significance in this area still await discovery. Conjectures of F. J. Dyson and M. L. Mehta (see Mehta, 1974, and 1967, chapter 4) provide added evidence for this view.

## 3.2 The Master Theorem

We begin with a small lemma that reduces the proof of the Master Theorem to a special case. By doing this, we facilitate the proof of the full result.

*Lemma.* If the Master Theorem is true for

$$A_{\binom{1}{1}} = A_{\binom{2}{2}} = \ldots = A_{\binom{n}{n}} = 0,$$

then it is true in general.

*Proof.* We begin by observing that the actual coefficient arising in the product of the linear forms in the Master Theorem may be obtained by use of the multinomial theorem; thus the coefficient is

$$\sum\nolimits^{*} p_1!p_2!\dots p_n!\prod_{i,j=1}^{n}\frac{A_{\binom{j}{i}}^{h_{ji}}}{h_{ji}!},$$

where $\sum^*$ is summed over all nonegative integers $h_{ji}$ subject to

$$\sum_{i=1}^{n} h_{ji} = p_j, \qquad \sum_{j=1}^{n} h_{ji} = p_i.$$

Hence the Master Theorem is equivalent to

$$(1) \qquad \sum_{p_1\geqq 0,\dots,p_n\geqq 0} p_1!p_2!\dots p_n! \sum\nolimits^{*}\prod_{i,j=1}^{n}\frac{A_{\binom{j}{i}}^{h_{ji}}}{h_{ji}} X_1^{p_1}X_2^{p_2}\dots X_n^{p_n}$$

$$= (\det(\delta_{ij} - A_{\binom{i}{j}}X_j))^{-1}.$$

Our object is to prove that the truth of (1) when

$$A_{\binom{1}{1}} = A_{\binom{2}{2}} = A_{\binom{3}{3}} = \dots A_{\binom{n}{n}} = 0$$

implies the truth of (1) in general.

We proceed to treat the sum in (1) by replacing each $p_j$ by $\sum_{i=1}^{n} h_{ji}$, and then we sum on each $h_{jj}$, using the binomial series. Therefore,

$$\sum_{p_1\geqq 0,\dots,p_n\geqq 0} p_1!p_2!\dots p_n! \sum\nolimits^{*}\prod_{i,j=1}^{n}\frac{A_{\binom{j}{i}}^{h_{ji}}}{h_{ji}!} X_1^{p_1}X_2^{p_2}\dots X_n^{p_n}$$

$$= \sum\nolimits^{\dagger}\frac{(h_{11}+h_{12}+\dots+h_{1n})!\dots(h_{n1}+h_{n2}+\dots+h_{nn})!}{\prod_{i,j=1}^{n} h_{ji}!}$$

$$\times\prod_{i,j=1}^{n} A_{\binom{j}{i}}^{h_{ji}} X_1^{h_{11}+\dots+h_{1n}}\dots X_n^{h_{n1}+\dots+h_{nn}}$$

$$= \sum\nolimits^{\#}\frac{(h_{12}+\dots+h_{1n})!\,(h_{21}+h_{23}+\dots+h_{2n})!\dots(h_{n1}+\dots+h_{n\,n-1})!}{\prod_{\substack{i,j=1\\ i\neq j}}^{n} h_{ji}!}$$

$$\times\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n} A_{\binom{j}{i}}^{h_{ji}}\right\} X_1^{h_{12}+\dots+h_{1n}}\dots X_n^{h_{n1}+\dots+h_{nn-1}}$$

$$\times\ (1 - A_{\binom{1}{1}} X_1)^{-h_{12}-\ldots-h_{1n}-1} \ldots (1 - A_{\binom{n}{n}} X_n)^{-h_{n1}-\ldots-h_{nn-1}-1}$$

$$= (1 - A_{\binom{1}{1}} X_1)^{-1}(1 - A_{\binom{2}{2}} X_2)^{-1} \ldots (1 - A_{\binom{n}{n}} X_n)^{-1}$$

$$\times \begin{vmatrix} 1 & -A_{\binom{1}{2}}\left(\dfrac{X_2}{1 - A_{\binom{2}{2}} X_2}\right) \ldots & -A_{\binom{1}{n}}\left(\dfrac{X_n}{1 - A_{\binom{n}{n}} X_n}\right) \\ -A_{\binom{2}{1}}\left(\dfrac{X_1}{1 - A_{\binom{1}{1}} X_1}\right) & 1 \qquad \ldots & -A_{\binom{2}{n}}\left(\dfrac{X_n}{1 - A_{\binom{n}{n}} X_n}\right) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ -A_{\binom{n}{1}}\left(\dfrac{X_1}{1 - A_{\binom{1}{1}} X_1}\right) & -A_{\binom{n}{2}}\left(\dfrac{X_2}{1 - A_{22} X_2}\right) \ldots & 1 \end{vmatrix}^{-1}$$

$$= (\det(\delta_{ij} - A_{\binom{i}{j}} X_j))^{-1}.$$

In the above, $\sum^{\dagger}$ is summed over all nonnegative integers $h_{ji}$ subject to $\sum_{i=1}^{n} h_{ji} = \sum_{i=1}^{n} h_{ij}$; $\sum^{\#}$ is the same as $\sum^{\dagger}$ with the added proviso that each $h_{jj} = 0$. The penultimate equation above follows from (1) in the special case in which all the diagonal coefficients are zero. Hence our lemma is established.

*Proof of the Master Theorem.* Let us consider the set $P$ of permutations without fixed points of the multiset $1^{a_1} 2^{a_2} \ldots r^{a_r}$ (a multiset is a set in which repetitions may occur; e.g., in the case under consideration, 1 appears $a_1$ times, 2 appears $a_2$ times, etc.). We shall denote such a permutation by using the standard notation

$$\begin{pmatrix} 1\ 1\ 1 & \ldots & 1 & 2 & \ldots\ldots & 2 & \ldots\ldots & r & \ldots\ldots\ldots\ldots & r \\ i_1 i_2 i_3 & \ldots & i_{a_1} & i_{a_1+1} & \ldots & i_{a_1+a_2} & \ldots & i_{a_1+\ldots+a_{r-1}+1} & \ldots & i_{a_1+\ldots+a_r} \end{pmatrix},$$

where each entry in the lower line is the image of the corresponding entry in the upper line under the permutation. When we say that the permutation is without fixed points, we mean that the upper entry is never equal to the corresponding lower entry.

We now define a semi-group structure on $P$ where the operation involved is the juxtaposition (or intercalation) of permutations. For example,

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ 2 & 2 & 1 & 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}.$$

Note that the above procedure is exactly like that used in the treatment of permutations of sets, except that columns with the same upper elements, such as $\substack{2\\1}\,\substack{2\\3}\,\substack{2\\4}$, must retain their left-to-right order in the factorization. We observe that primes in this semi-group are precisely the cycles of length at least 2 (since there are no fixed points), and each permutation factors uniquely (up to permisseble commutations) into primes.

With these facts in mind, we replace each $A_{\binom{i}{j}}$ by the symbol $\binom{i}{j}$. Recalling the lemma, we assume that $\binom{1}{1} = \ldots = \binom{n}{n} = 0$. Then the desired coefficient in the product of linear forms is nothing but

$$\sum \begin{pmatrix} i_1 i_2 \ldots i_N \\ j_1 j_2 \ldots j_N \end{pmatrix},$$

where the sum is extended over all permutations without fixed points of the multiset $1^{p_1} 2^{p_2} \ldots n^{p_n}$. If we multiply this sum by $X_1^{p_1} X_2^{p_2} \ldots X_n^{p_n}$ and sum over all $p_i \geqq 0$, we obtain the generating function for all multiset permutations without fixed points on the first $n$ positive integers:

$$G = \sum \begin{pmatrix} i_1 i_2 \ldots i_N \\ j_1 j_2 \ldots j_N \end{pmatrix} X_{j_1} X_{j_2} \ldots X_{j_N}.$$

On the other hand,

$$D = \begin{vmatrix} 1 & -\binom{1}{2} X_2 & \ldots & -\binom{1}{n} X_n \\ -\binom{2}{1} X_1 & 1 & \ldots & -\binom{2}{n} X_n \\ \vdots & \vdots & & \vdots \\ -\binom{n}{1} X_1 & -\binom{n}{2} X_2 & \ldots & 1 \end{vmatrix},$$

when expanded, equals

$$\sum \begin{pmatrix} i_1 i_2 \ldots i_M \\ j_1 j_2 \ldots j_M \end{pmatrix} (-1)^c X_{j_1} X_{j_2} \ldots X_{j_M},$$

where $\begin{pmatrix} i_1 i_2 \dots i_M \\ j_1 j_2 \dots j_M \end{pmatrix}$ is a permutation without fixed points of some subset of 1, 2, 3, ..., $n$, and $c$ is the number of cycles in this permutation.

To conclude our proof of the Master Theorem, we need only show that $D \cdot G = 1$. Now

$$(2) \qquad D \cdot G = \sum \left( \sum_{\pi_1 \left| \left( \begin{smallmatrix} i_1 i_2 \dots i_N \\ j_1 j_2 \dots j_N \end{smallmatrix} \right) \right.} (-1)^{c(\pi_1)} \right) \begin{pmatrix} i_1 \dots i_N \\ j_1 \dots j_N \end{pmatrix} X_{j_1} X_{j_2} \dots X_{j_N},$$

where the outer sum runs over all multiset permutations $\begin{pmatrix} i_1 i_2 \dots i_N \\ j_1 j_2 \dots j_N \end{pmatrix}$ without fixed points on the first $n$ positive integers, and the inner sum runs over all left factors $\pi_1$ of $\begin{pmatrix} i_1 i_2 \dots i_N \\ j_1 j_2 \dots j_N \end{pmatrix}$ made up of disjoint cycles. If the left prime factors of $\begin{pmatrix} i_1 i_2 \dots i_N \\ j_1 j_2 \dots j_N \end{pmatrix}$ are denoted by $\sigma_1, \sigma_2, \dots, \sigma_r$, and if we define

$$\mu(\pi) = \begin{cases} (-1)^{c(\pi)} \text{ if } \pi \text{ factors into } c(\pi) \text{ disjoint cycles,} \\ 0 \text{ otherwise,} \end{cases}$$

then

$$\sum_{\pi_1 \left| \left( \begin{smallmatrix} i_1 \dots i_N \\ j_1 \dots j_N \end{smallmatrix} \right) \right.} (-1)^{c(\pi_1)} = \prod_{j=1}^{r} (1 + \mu(\sigma_j)) = \begin{cases} 1 \text{ if } r = 0, \\ 0 \text{ otherwise.} \end{cases}$$

Thus the only term on the right side of (2) that does not vanish corresponds to the empty permutation of the empty set. Therefore $D \cdot G = 1$, and the Master Theorem is proved.

### 3.3 Good's Proof of Dyson's Conjecture

Let us examine for a moment a special case of the Master Theorem, namely,

$$A_{\binom{1}{1}} = A_{\binom{2}{2}} = A_{\binom{3}{3}} = 0, \qquad A_{\binom{1}{2}} = A_{\binom{2}{3}} = A_{\binom{3}{1}} = 1,$$
$$A_{\binom{1}{3}} = A_{\binom{2}{1}} = A_{\binom{3}{2}} = -1.$$

Thus the coefficient of

$$X_1^{a_2+a_3} X_2^{a_1+a_3} X_3^{a_1+a_2}$$

in

$$(X_2 - X_3)^{a_2+a_3}(X_3 - X_1)^{a_1+a_3}(X_1 - X_2)^{a_1+a_2}$$

is identical with the same coefficient in

$$\begin{vmatrix} 1 & -X_2 & X_3 \\ X_1 & 1 & -X_3 \\ -X_1 & X_2 & 1 \end{vmatrix}^{-1} = (1 + X_2X_3 + X_1X_2 + X_1X_3)^{-1}$$

$$= \sum_{m \geqq 0} (-1)^m \sum_{j_1+j_2+j_3=m} \frac{m!}{j_1!j_2!j_3!} X_1^{j_2+j_3} X_2^{j_1+j_3} X_3^{j_1+j_2}.$$

Hence the desired coefficient arises from the lone term $j_1 = a_1$, $j_2 = a_2$, $j_3 = a_3$, and is therefore

$$(-1)^{a_1+a_2+a_3}(a_1 + a_2 + a_3)!/a_1!a_2!a_3!.$$

D. Foata points out that this result may be stated in the following, slightly altered form as a corollary of the Master Theorem (see [61]).

*Corollary.* The constant term in the expansion of $(1 - X_3/X_2)^{a_3}(1 - X_2/X_3)^{a_2}(1 - X_1/X_3)^{a_1}(1 - X_3/X_1)^{a_3}(1 - X_2/X_1)^{a_2}(1 - X_1/X_2)^{a_1}$ is $(a_1 + a_2 + a_3)!/a_1!a_2!a_3!$.

This result is actually Dixon's theorem on the summability of the well-poised ${}_3F_2$ (see W. N. Bailey, Generalized Hypergeometric Series, Cambridge University Press, 1935, p.13), and in the case $a_1 = a_2 = a_3 = p$, it reduces to Dixon's theorem concerning the sum of the cubes of the binomial coefficients.

Interestingly enough, this corollary has a generalization which seems to be related to the Master Theorem and suggests the existence of more general results in this area:

*Theorem (Dyson's Conjecture).* The constant term in the expansion of

$$\prod_{1 \leqq i \neq j \leqq n} (1 - X_j/X_i)^{a_j}$$

is $(a_1 + a_2 + \ldots + a_n)!/a_1!a_2! \ldots a_n!$.

*Remark.* F. J. Dyson (1962) conjectured this result in an extensive work on statistical mechanics. Shortly thereafter, J. Gunson (1962) and K. Wilson (1962) proved this result. D. Foata pointed out the intersection of Dyson's conjecture and MacMahon's Master Theorem. I. J. Good (1970) gave a very short and elegant proof which we now present.

*Proof.* We begin by noting that

$$1 = \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}, \tag{3}$$

since the right side of (3) is a polynomial of degree at most $n - 1$ that assumes the value 1 for $n$ different values of $x$, and is therefore identically 1. Setting $x = 0$ in (3), we obtain

$$1 = \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{(1 - x_k/x_j)}. \tag{4}$$

Let us write

$$F(x_1, \ldots, x_n; a_1, \ldots, a_n) = \prod_{1 \leq i \neq j \leq n} (1 - x_j/x_i)^{a_j}, \tag{5}$$

and observe that if we multiply equation (4) by this function, then when no $a_j = 0$,

$$F(x_1, \ldots, x_n; a_1, \ldots, a_n) = \sum_{j=1}^{n} F(x_1, \ldots, x_n; a_1, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots, a_n).$$

Consequently, if $G(a_1, \ldots, a_n)$ denotes the constant term in (5), then when no $a_j = 0$,

$$G(a_1, \ldots, a_n) = \sum_{j=1}^{n} G(a_1, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots, a_n). \tag{6}$$

If a particular $a_j = 0$, then $x_j$ occurs to nonpositive powers only in (5), so that $G(a_1, \ldots, a_n)$ is equal to the constant term in

$$F(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n; a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n),$$

i.e.,

$$G(a_1, \ldots, a_n) = G(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n) \text{ if } a_j = 0. \tag{7}$$

Finally,

$$G(0, \ldots, 0) = 1. \tag{8}$$

Now equations (6), (7), and (8) uniquely define $G(a_1, \ldots, a_n)$, and since $(a_1 + a_2 + \ldots + a_n)!/a_1!a_2! \ldots a_n!$ also satisfies (6), (7), and (8), we must have

$$G(a_1, \ldots, a_n) = (a_1 + a_2 + \ldots + a_n)!/a_1!a_2! \ldots a_n!.$$

This concludes our proof.

## 3.4 References


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## 3.5 Summaries of the Papers

[45] A certain class of generating functions in the theory of numbers, *Phil. Trans.*, *185* (1894), 111–160.

This is the extremely important paper in which MacMahon introduces and proves his celebrated "Master Theorem." (See section 3.1. for a statement of the theorem.)

In section 1 MacMahon proves the Master Theorem. In section 2 he applies it to several problems in the theory of permutations, including two problems that appeared earlier (in [46]); the latter presumably led to his discovery of the Master Theorem. In sections 3–6 he considers when a form $V_n$ linear in $x_1, \ldots, x_n$ can be written in the form $\det(\delta_{ij} - a_{ij}x_j)$. The paper concludes with a further exploration of permutation problems.

[61] The sums of powers of the binomial coefficients, *Quart. J. Math.*, *33* (1902), 274–288.

This paper applies MacMahon's Master Theorem to summations of binomial coefficients. MacMahon proves (among many other results) that

$$\sum_{i=0}^{p} \binom{p}{i}\binom{q}{i} = \binom{p+q}{p} \qquad \text{(the Chu-Vandermonde sum),}$$

and,

$$\sum_{i=0}^{p}(-1)^i\binom{2p}{i}^3 = (-1)^p(2p)!/(p!)^3 \qquad \text{(Dixon's summation).}$$

Use of the Master Theorem makes the derivation of these and many other results extremely elegant (see section 3.3).

In this way MacMahon treats the sum

$$\sum_{i=0}^{p}\binom{p}{i}^{\alpha}$$

for $\alpha = 2, 3,$ or $4$.

# Reprints of the Papers