HW1.

3. Give *combinatorial* proofs of the following identities, where x, y, n, a, b are nonnegative integers.

(g) [2+] $\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{j=0}^{n} \binom{n}{j}\binom{2n-j}{n}(x-1)^j$

Answer:

The left-hand side counts the number of triples (S, T, f), where $S \subseteq [n]$, $T \subseteq [n+1, 2n]$, $\#S = \#T$, and $f\colon S \to [x]$. The right-hand side counts the number of triples (A, B, g), where $A \subseteq [n]$, $B \in \binom{[2n]-A}{n}$, and $g : A \to [x-1]$. Given (S, T, f), define (A, B, g) as follows: $A = f^{-1}([x-1])$, $B = ([n] - S) \cup T$, and $g(i) = f(i)$ for $i \in [x-1]$.

Outline:

1. Let $\mathbb{X} = \left\{ (S, T, f) \;\middle|\; \begin{array}{l} S \subseteq [n],\ T \subseteq [n+1, 2n], \\ |S| = |T|,\ f : S \to [x] \end{array} \right\}$

   Show $|\mathbb{X}| = \sum_{k=0}^{n} \binom{n}{k}^2 x^k$

2. Let $\mathbb{Y} = \left\{ (A, B, g) \;\middle|\; \begin{array}{l} A \subseteq [n],\ B \subseteq [2n] \setminus A \\ |B| = n,\ g : A \to [x-1] \end{array} \right\}$

   Show $|\mathbb{Y}| = \sum_{j=0}^{n} \binom{n}{j}\binom{2n-j}{n}(x-1)^j$

3. Construct

   (i) $\varphi : \mathbb{X} \to \mathbb{Y}$, where φ : 1-1

   (ii) $\psi : \mathbb{Y} \to \mathbb{X}$, where ψ : 1-1

   $\Rightarrow |\mathbb{X}| = |\mathbb{Y}|$

1. $$\boxed{\begin{array}{l} \text{Let } X = \left\{ (S, T, f) \;\middle|\; \begin{array}{l} S \subseteq [n],\ T \subseteq [n+1, 2n], \\ |S| = |T|,\ f : S \to [x] \end{array} \right\} \\[2ex] \text{Show } |X| = \displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 x^k \end{array}}$$

$$\underset{\#k}{\underset{\downarrow}{S}} \subseteq \underset{\#n}{\underset{\downarrow}{[n]}},\ \underset{\#k}{\underset{\downarrow}{T}} \subseteq \underset{\#n}{\underset{\downarrow}{[n+1, 2n]}},\ |S| \underset{\text{say } \#k}{\underset{\downarrow}{=}} |T|,\ f : \underset{\#k}{\underset{\downarrow}{S}} \to \underset{\#x}{\underset{\downarrow}{[x]}}$$

$$|X| = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\text{pick } S} \underbrace{\binom{n}{k}}_{\text{pick } T} \underbrace{x^k}_{\text{pick } f}$$

2. $$\boxed{\begin{array}{l} \text{Let } Y = \left\{ (A, B, g) \;\middle|\; \begin{array}{l} A \subseteq [n],\ B \subseteq [2n] \setminus A \\ |B| = n,\ g : A \longrightarrow [x-1] \end{array} \right\} \\[2ex] \text{Show } |Y| = \displaystyle\sum_{j=0}^{n} \binom{n}{j} \binom{2n-j}{n} (x-1)^j \end{array}}$$

$$\underset{\#j}{\underset{\downarrow}{A}} \subseteq \underset{\#n}{\underset{\downarrow}{[n]}},\ \underset{\#n}{\underset{\downarrow}{B}} \subseteq \underbrace{[2n] \setminus A}_{\#2n-j},\ |B| = n,\ g : \underset{\#j}{\underset{\downarrow}{A}} \longrightarrow \underset{\#x-1}{\underset{\downarrow}{[x-1]}}$$

$$|Y| = \sum_{j=0}^{n} \underbrace{\binom{n}{j}}_{\text{pick } A} \underbrace{\binom{2n-j}{n}}_{\text{pick } B} \underbrace{(x-1)^j}_{\text{pick } g}$$

3.(i) $\boxed{\text{Construct } \varphi : \mathbb{X} \to \mathbb{Y} \text{ , where } \varphi : 1\text{-}1}$

Let φ define on $\mathbb{X}$ and $\varphi(S, T, f) = (A', B', g')$

where $A' = f^{-1}([x-1])$, $B' = ([n]-S) \cup T$

and $g : A \to [x-1]$ s.t. $g(i) = f(i)$ for $i \in [x-1]$

Hence, $A' \subseteq S \subseteq [n]$,

$\because A' \cap B' = \phi \quad \therefore B' \subseteq [2n] \setminus A$

$\because [n] \cap T = \phi \quad \therefore |B'| = |[n]| - |S| + |T| = n$

$\because A' = f^{-1}([x-1]) \quad \therefore g$ is well-defined

Therefore $(A', B', g) \in \mathbb{Y}$, Image$(\varphi) \subset \mathbb{Y}$ and $\varphi : 1\text{-}1$

3.(ii) $\boxed{\text{Construct } \psi : \mathbb{Y} \to \mathbb{X} \text{ , where } \psi : 1\text{-}1}$

Let ψ define on $\mathbb{Y}$ and $\psi(A, B, g) = (S', T', f')$

where $T' = B \cap [n+1, 2n]$, $S' = [n] \cap B^{c}$ ← 補集

$\therefore S' \subseteq [n]$, $T' \subseteq [n+1, 2n]$ $\because |B| = n$ $\therefore |T'| = |S'|$

$\because A \cap B = \phi$ and $A \subset [n]$ $\therefore A \subseteq S'$

let $f : S' \longrightarrow [x]$, where $f(i) = \begin{cases} g(i) & \text{if } i \in A \\ x & \text{if } i \notin A \end{cases}$

Therefore $(S', T', f') \in \mathbb{X}$, Image$(\psi) \subseteq \mathbb{X}$ and $\psi : 1\text{-}1$