HW 2

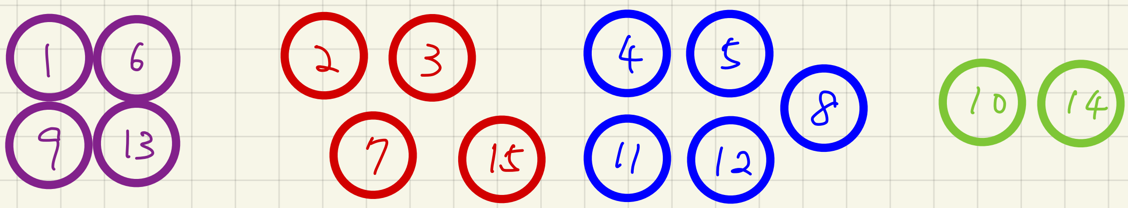
45. [2+] Let $S(n,k)$ denote a Stirling number of the second kind. The generating function $\sum_n S(n,k)x^n = x^k/(1-x)(1-2x)\cdots(1-kx)$ implies the identity

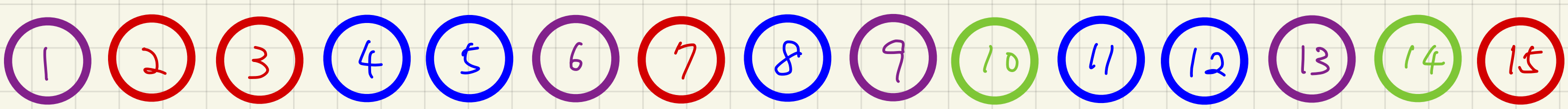
$$S(n,k) = \sum 1^{a_1-1}2^{a_2-1}\cdots k^{a_k-1}, \tag{1.123}$$

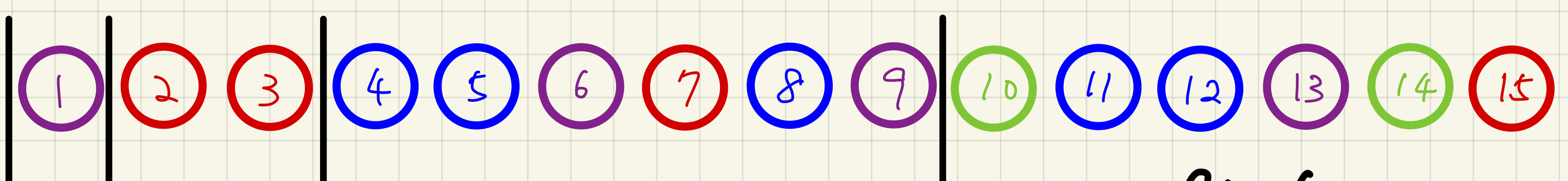
the sum being over all compositions $a_1 + \cdots + a_k = n$. Give a *combinatorial* proof of (1.123) analogous to the second proof of Proposition 1.3.7. That is, we want to associate with each partition π of $[n]$ into k blocks a composition $a_1 + \cdots + a_k = n$ such that exactly $1^{a_1-1}2^{a_2-1}\cdots k^{a_k-1}$ partitions π are associated with this composition.

45. Define $a_{i+1} + a_{i+2} + \cdots + a_k$ to be the least r such that when $1, 2, \ldots, r$ are removed from π, the resulting partition has i blocks.

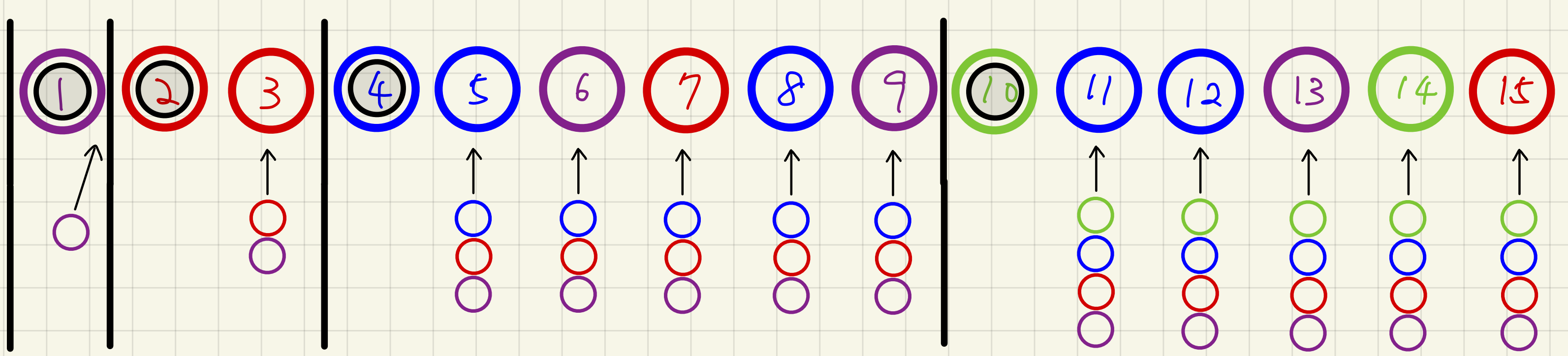
$n = 15,\ k = 4$







$a_1 = 1$ $a_2 = 2$ $a_3 = 6$ $a_4 = 6$



1^{a_1-1} 2^{a_2-1} 3^{a_3-1} 4^{a_4-1}

$$\because \sum_i S(i,k)\, x^i = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$

$$= \underset{\underset{[x^k]}{\uparrow}}{x^k}\,(1+x+\underset{\underset{[x^{a_1-1}]}{\uparrow}}{x^2}+\dots)(1+2x+\underset{\underset{[x^{a_2-1}]}{\uparrow}}{2^2x^2}+\dots)\underset{\cdots}{\cdots}(1+\underset{\underset{[x^{a_k-1}]}{\uparrow}}{kx}+k^2x^2+\dots)$$

$$\therefore S(n,k) = [x^n] \sum_i S(i,k)\, x^i$$

$$= \sum_{a_1+\dots+a_k=n} \left([x^k]\, x^k\right)\left([x^{a_1-1}]\,(1+x+x^2+\dots)\right)\cdots\left([x^{a_k-1}]\,(1+kx+k^2x^2+\dots)\right)$$

$$= \sum_{a_1+\dots+a_k=n} 1^{a_1-1}\, 2^{a_2-1} \cdots k^{a_k-1}$$