數學二離散數學 2022 秋, 第一次期中考 解答

學號: ________________________, 姓名: ________________________

本次考試共有 11 頁 (包含封面)，有 10 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Find integers a and b such that

$$m^2 = a\binom{m}{2} + b\binom{m}{1}$$

for all m. Then use it to find the sum the series $1^2 + 2^2 + ... + n^2$.

Answer: $a =$ **2** , $b =$ **1** , $1^2 + 2^2 + ... + n^2 =$ $2\binom{n+1}{3} + \binom{n+1}{2} = \frac{n(n+1)(2n+1)}{6}$.

Solution：

$$m^2 = a\binom{m}{2} + b\binom{m}{1} = a\frac{m(m-1)}{2} + bm = \frac{a}{2}m^2 + \frac{-a+2b}{2}m$$

Thus $a = 2, b = 1$

Recall the equation (5.19) from the textbook, we have

$$\binom{n+1}{k+1} = \binom{0}{k} + \binom{1}{k} + \binom{2}{k} + ... + \binom{n-1}{k} + \binom{n}{k}$$

Therefore,

$$\begin{aligned} 1^2 + 2^2 + ... + n^2 &= \sum_{m=1}^{n}\left[2\binom{m}{2} + \binom{m}{1}\right] = 2\left[\sum_{m=1}^{n}\binom{m}{2}\right] + \left[\sum_{m=1}^{n}\binom{m}{1}\right] \\ &= 2\binom{n+1}{3} + \binom{n+1}{2} \\ &= 2\frac{(n+1)n(n-1)}{6} + \frac{(n+1)n}{2} = n(n+1)\left[\frac{n-1}{3} + \frac{1}{2}\right] \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

2. (10 points) A bakery sells six different kinds of donuts, includes original, glazed, chocolated, banana milk, mango and strawberry. If the bakery has at least a dozen of each kind, how many different options for a dozen of donuts are there? What if a box is to contain at least three of chocolated donuts and two strawberry donuts?

一家麵包店銷售六種不同的甜甜圈，口味是原味、糖霜、巧克力、香蕉牛奶、芒果跟草莓。如果該店每種糕點至少有一打（12 個），那麼 (a) 可能配製成多少打不同種類的甜甜圈？(b) 如果盒子裡至少要有三個巧克力口味和兩個草莓口味的，又是多少種？

Answer: (a) $\binom{17}{5}$, (b) $\binom{12}{5}$.

Solution：

(a) Suppose that $1 \leq i \leq 6$, we choose n_i donuts of the i^{th} kind. Each of $\{n_i\}_{i=1}^{6}$ is nonnegative and $\sum_{i=1}^{6} n_i = 12$. The number of ways to pick the $\{n_i\}_{i=1}^{6}$ is

$$\binom{12+6-1}{6-1} = \binom{17}{5}$$

(b) Since chocolated donuts are the third kind and strawberry donuts are the sixth kind, we define $m_i = n_i$ for $i = 1, 2, 4, 5$, $m_3 = n_3 - 3$, $m_6 = n_6 - 2$. Each of $\{m_i\}_{i=1}^{6}$ is nonnegative and $\sum_{i=1}^{6} m_i = 12 - 3 - 2 = 7$. The number of ways to pick the $\{m_i\}_{i=1}^{6}$ is

$$\binom{7+6-1}{6-1} = \binom{12}{5}$$

3. (10 points) Twenty-four different books are to be put on six book shelves, each of which holds at least twenty-four books. 要將 24 本不同的書放到 6 個書架上，每個書架至少能放 24 本。

(a) How many different arrangements are there if you only care about the number of books on the shelves (and not which book is where)? 如果只關心書架上書的數量（而不關心哪本書在哪個書架上），那麼有多少種不同的擺法？

(b) How many different arrangements are there if you care about which books are where, but the order of the books on the shelves doesn't matter? 如果關心哪本書放在哪個書架上，但不關心書在書架上的順序，那麼有多少種不同的擺法？

(c) How many different arrangements are there if the order on the shelves does matter? 如果需要考慮書在書架上的順序，那麼又有多少種不同的擺法？

Answer: (a) $\binom{29}{5}$, (b) 6^{24}, (c) $24!\binom{29}{5}$.

Solution：

(a) For $1 \le i \le 6$, let x_i denote the number of books on shelf i. The seeking number are the number of integral solution for the following equation.

$$\sum_{i=1}^{6} x_i = 24,\ x_i \ge 0,\ \text{for } i = 1\ 6$$

The answer is

$$\binom{24+6-1}{6-1} = \binom{29}{5}$$

(b) For $1 \le i \le 6$, there's 6 ways to put book i on a shelf. The answer is 6^{24}.

(c) For $1 \le i \le 6$, let x_i denote the number of books on shelf i. Using part (a), there's $\binom{29}{5}$ to find the number of books on each shelf.

We can divides books match the above number in $\binom{24}{x_1, x_2, x_3, x_4, x_5, x_6}$ ways. For each shelf i, we can order the book in $x_i!$ ways.

Thus the answer is product all above:

$$\binom{29}{5}\binom{24}{x_1, x_2, x_3, x_4, x_5, x_6} x_1!x_2!x_3!x_4!x_5!x_6! = 24!\binom{29}{5}$$

4. (10 points) A student has 37 days to prepare for an examination. From past experience she knows that she will require no more than 60 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day, however), there is a succession of days during which she will have studied exactly 13 hours.

一個學生有 37 天用來準備考試。根據以往的經驗，她知道她的唸書時間不超過 60 小時，她也希望她每天至少要唸一個小時的書。證明：無論她怎麼安排她的讀書計畫（每天的唸書時間都是一個整數小時)，都存在連續的某幾天，在這幾天裡，她恰好唸了 13 個小時的書。

Solution：

For $1 \leq i \leq 37$ let b_i denote the number of hours the student studies from day 1 to day i. We have $1 \leq b_1 < b_2 < ... < b_{37} \leq 60$ since $\sum_{i=1}^{37} b_i \leq 60$. Let $c_i = b_i + 13$. Consider the numbers $\{c_1, c_2, ..., c_{37}\} \cup \{b_1, b_2, ..., b_{37}\}$. There are 74 numbers in the list, all among $1, 2, ..., 73$ since $\sum_{i=1}^{37} c_i < 60+13 = 73$. By the pigeonhole principle the numbers $\{c_1, c_2, ..., c_{37}\} \cup \{b_1, b_2, ..., b_{37}\}$ are not distinct. Therefore there exist integers i, j $(0 \leq i < j \leq 37)$ such that $b_i = c_j$. Therefore $b_{i+1} + \cdot \cdot \cdot + b_j = 13$. During the days $i+1, . . . ,j$ the student will have studied exactly 13 hours.

5. (10 points) Construct a permutation whose inversion sequences are 3, 2, 1, 2, 0, 0, 2, 0, 0.

Answer: **5 3 2 1 6 4 8 9 7** .

Solution：

By Algorithm I:

9	.... $a_9 = 0$
8 9	.... $a_8 = 0$
8 9 7	.... $a_7 = 2$
6 8 9 7	.... $a_6 = 0$
5 6 8 9 7	.... $a_5 = 0$
5 6 4 8 9 7	.... $a_4 = 2$
5 3 6 4 8 9 7	.... $a_3 = 1$
5 3 2 6 4 8 9 7	.... $a_2 = 2$
5 3 2 1 6 4 8 9 7	.... $a_1 = 3$

Solution：

By Algorithm II:

-	-	-	1	-	-	-	-	-	.... $a_1 = 3$
-	-	2	1	-	-	-	-	-	.... $a_2 = 2$
-	3	2	1	-	-	-	-	-	.... $a_3 = 1$
-	3	2	1	-	4	-	-	-	.... $a_4 = 2$
5	3	2	1	-	4	-	-	-	.... $a_5 = 0$
5	3	2	1	6	4	-	-	-	.... $a_6 = 0$
5	3	2	1	6	4	-	-	7	.... $a_7 = 2$
5	3	2	1	6	4	8	-	7	.... $a_8 = 0$
5	3	2	1	6	4	8	9	7	.... $a_9 = 0$

6. (10 points) How many permutations of 1, 2, 3, 4, 5, 6, 7 have (a) exactly 23 inversions? (b) exactly 21 inversions? (c) exactly 19 inversions?

Answer: (a) **none** , (b) **1** , (c) **20** .

Solution：

(a)(b) For a permutation of 1, 2, 3, 4, 5, 6, 7 the corresponding inversion sequence $(a_1, a_2, a_3, a_4, a_5, a_6, a_7$ satisfies $0 \leq a_i \leq 7 - i$ for $1 \leq i \leq 7$. The total number os inversions is $\sum_{i=1}^{7} a_i$. This total is at most $6+5+4+3+2+1+0 = 21$ with equality if and only if $a_i = 7 - i$. Therefore there is NO permutation with 23 inversions and there is only ONE permutation with 21 inversions.

(c) The number of permutations with 19 inversions is equal to the number of integral solutions to

$$\sum_{i=1}^{7} a_i = 19, \text{with } 0 \leq a_i \leq 7 - i \ (1 \leq i \leq 7)$$

Each solution $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ is obtained from (6, 5, 4, 3, 2, 1, 0) by subtracting 1 from two of the first 6 coordinates ($\binom{6}{2}$ ways) or subtracting 2 from one of the first four coordinates (5 ways). Therefore the number of solutions is $\binom{6}{2} + 5 = 20$. There are 20 permutations with 19 inversions.

7. (10 points) Determine the coefficient of $x_1^3x_2x_3^4x_5^2$ in the expansion of
在下列展開式中找出 $x_1^3x_2x_3^4x_5^2$ 這項的係數。

(a) $(x_1 + x_2 + x_3 + x_4 + x_5)^{10}$

(b) $(x_1 - x_2 + 3x_3 + x_4 - 2x_5)^{10}$

Answer: (a) $\dfrac{10!}{3! \times 1! \times 4! \times 2!} = 12600$, (b) $\dfrac{(-9) \times 10!}{8}$.

Solution：

(a)

$$\binom{10}{3,1,4,2} = \frac{10!}{3! \times 1! \times 4! \times 2!} = 12600$$

(b)

$$\begin{aligned}
&\binom{10}{3,1,4,2} \times 1^2 \times (-1) \times 3^4 \times 1^0 \times (-2)^2 \\
=&\frac{10! \times 1^2 \times (-1) \times 3^4 \times 1^0 \times (-2)^2}{3! \times 1! \times 4! \times 2!} \\
=&\frac{(-9) \times 10!}{8}
\end{aligned}$$

8. (10 points) Prove that if 11 integers are selected from among 1,2, ...,20, then the selection includes integers a and b such that $a-b=2$.

   證明從 1,2, ...,20 中選定 11 個數字，則必定可以從選定的數中找出兩個數 a 跟 b 使得 $a-b=2$。

   Solution：

   Let 's divides 1 20 into 10 groups as the following:

   (1, 3), (2, 4), (5, 7), (6, 8), (9, 11), (10, 12), (13, 15), (14, 16), (17, 19), (18, 20)

   By the pigeonhole principle, the selected 11 integer must contains two integers a, b are in a same group. Thus $|a-b|=2$.

9. (10 points) Use combinatorial reasoning to prove the identity

$$\binom{n}{n_1, n_2, ..., n_k} = \binom{n-1}{n_1 - 1, n_2, ..., n_k} + \binom{n-1}{n_1, n_2 - 1, ..., n_k} + ... + \binom{n-1}{n_1, n_2, ..., n_k - 1}$$

Solution：

我上課有證過。

10. (10 points) Find the Ramsey number $R(5,2)$ and explain why.

Answer: **5** .

Solution：

The following are copied from textbook page 79:

The Ramsey number $r(2,n)$ and $r(m,2)$ are easy to determine. We show that $r(2,n)=n$:

1. $r(2,n) \le n$: If we color the edges of K_n either red or blue, them either some edge is colored red (and so we have a red K_2) or all edges are blue (and so we have a blue K_n).

2. $r(2,n) > n-1$: If we color all edges of K_{n-1} blue, then we have either a red k_2 nor a blue K_n.

In a similar way, we show the $r(m,2)=m$. The numbers $r(2,n)$ and $r(m,2)$ with $m,n \ge 2$ are *trivial Ramsey numbers.*

Thus, we can prove it by replace m with 5.

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											