數學二離散數學 2022 秋, 第二次期中考 解答

學號: ________________________________, 姓名: ________________________________

本次考試共有 11 頁 (包含封面)，有 10 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (15 points) How many ways can n couples (2n people) be seated in a line so that forno couple do the two people of the couple sit together. (All couples mustbe split up.)

   有多少種方式可以讓 n 對夫妻（2n 人）排成一排，使得沒有一對夫妻中的兩個人坐在一起。(所有夫妻都必須分開。)

   Answer: $n!\sum_{i=0}^{n}(-2)^i\frac{(2n-i)!}{(n-i)!i!}$.

   Solution：

   We use inclusion-exclusion. Let S be the set of $(2n)!$ arrangements. Let A_i be the set of arrangements in which couple i sits together. Then a_I counts the arrangements in which couple i sits together for each $i \in I$. To count this we gave $(2n{-}i)$ units to arrange (i couples and $2n{-}2i$ other individuals). Once they are arranged each of the i couple can be arranged in two ways. So $a_I = 2^i(2n{-}i)!$ if $|I| = i$. We count

$$
\begin{aligned}
|\cap_{i=1}^{n}\overline{A_i}| &= \sum_{i=0}^{n}(-1)^i\sum_{I\in 2^S}a_I \\
&= \sum_{i=0}^{n}(-1)^i\binom{n}{i}2^i(2n-i)! \\
&= n!\sum_{i=0}^{n}(-2)^i\frac{(2n-i)!}{(n-i)!i!}
\end{aligned}
$$

2. (10 points) Determine the number of integral solutions of the equation

$$x + y + z + w = 20$$

that satisfy

$$1 \le x \le 6,\ 0 \le y \le 7,\ 4 \le z \le 8,\ 2 \le w \le 6.$$

Answer: 96 .

Solution：

Let's change of variables $y_1 = x - 1$, $y_2 = y$, $y_3 = z - 4$, $y_4 = w - 2$.

The question become seeking the number of integral solutions to

$$(x-1) + (y) + (z-4) + (w-2) = y_1 + y_2 + y_3 + y_4 = 20 - 1 - 0 - 4 - 2 = 13$$

with

$$0 \le x - 1 = y_1 \le 5,\ 0 \le y_2 \le 7,\ 0 \le z - 4 = y_3 \le 4,\ 0 \le w - 2 = y_4 \le 4$$

Let S denote the set of nonnegative integral solutions to $y_1 + y_2 + y_3 + y_4 = 13$. Let A_1 (resp. A_2) (resp. A_3) (resp. A_4) denote the set of elements in S such that $y_1 \ge 6$ (resp. $y_2 \ge 8$) (resp. $y_3 \ge 5$) (resp. $y_4 \ge 5$). We seek $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$. We have

set	S	A_1	A_2	A_3	A_4
size	$\binom{16}{3}$	$\binom{10}{3}$	$\binom{8}{3}$	$\binom{11}{3}$	$\binom{11}{3}$

set	$A_1 \cap A_2$	$A_1 \cap A_3$	$A_1 \cap A_4$	$A_2 \cap A_3$	$A_2 \cap A_4$	$A_3 \cap A_4$	$A_i \cap A_j \cap A_k$
size	0	$\binom{5}{3}$	$\binom{5}{3}$	$\binom{3}{3}$	$\binom{3}{3}$	$\binom{6}{3}$	0

By inclusion-exclusion

$$\begin{aligned}
&|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| \\
&= \binom{16}{3} - \binom{10}{3} - \binom{8}{3} - \binom{11}{3} - \binom{11}{3} + \binom{5}{3} + \binom{5}{3} + \binom{3}{3} + \binom{3}{3} + \binom{6}{3} \\
&= 96
\end{aligned}$$

3. (10 points) Determine the number of ways to place six non-attacking rooks on the following 6-by-6 board, with forbidden positions as shown.

		x	x		
		x	x		
x	x				
x					

Answer: 184 .

Solution：

$$6! - 7 \times 5! + 15 \times 4! - 10 \times 3! + 2 \times 2! = 184$$

跟課本 page 180 的例子算法一模一樣，稍微平移位置而已。

4. (10 points) Solve the recurrence relation $h_n = 4h_{n-1} - 4h_{n-2}$ with initial values $h_0 = 3$ and $h_1 = 16$.

Answer: $h_n = 3 \times 2^n + 5 \times n2^n$.

Solution：

Rewrite the given recurrence relation as $h_n - 4h_{n-1} + 4h_{n-2} = 0$ The characteristic equation of the recurrence relation is $x^2 - 4x + 4 = (x-2)^2 = 0$. Thus we have the characteristic are $x = 2$.

Therefore,

$$h_n = c_1 \times 2^n + c_2 \times n2^n$$

Using the initial condition

$$h_0 = 3 = c_1 \times 2^0 + c_2 \times 0 \times 2^0$$
$$h_1 = 16 = c_1 \times 2^1 + c_2 \times 1 \times 2^1$$

.

We have

$$c_1 = 3, \; c_2 = 5$$

5. (10 points) Solve the recurrence relation $h_n = 4h_{n-2}$ with initial values $h_0 = 4$ and $h_1 = 1$.

Answer: $h_n = 2^{n+1+(-1)^n} = \frac{9}{4} \times 2^n + \frac{7}{4} \times (-2)^n$.

Solution：

By Method I:

$$h_{2n} = 4^1 h_{2n-2} = 4^2 h_{2n-2\times 2} = 4^3 h_{2n-2\times 3} = \ldots = 4^n h_{2n-2\times n} = 4^n h_0 = 4^{n+1} = 2^{2n+2}$$

$$h_{2n+1} = 4^1 h_{2n+1-2} = 4^2 h_{2n+1-2\times 2} = 4^3 h_{2n+1-2\times 3} = \ldots = 4^n h_{2n+1-2\times n} = 4^n h_1 = 4^n = 2^{2n}$$

Since $h_{2n} = 2^{2n+2}$, $h_{2n+1} = 2^{2n}$, we have $h_n = 2^{n+1+(-1)^n}$

Solution：

By Method II:

Rewrite the given recurrence relation as $h_n - 4h_{n-2} = 0$ The characteristic equation of the recurrence relation is $x^2 - 4 = (x-2)(x+2) = 0$. Thus we have the characteristic are $x = 2, -2$.

Therefore,

$$h_n = c_1 \times 2^n + c_2 \times (-2)^n$$

Using the initial condition

$$h_0 = 4 = c_1 \times 2^0 + c_2 \times (-2)^0$$
$$h_1 = 1 = c_1 \times 2^1 + c_2 \times (-2)^1$$

.

We have

$$c_1 = 9/4, \; c_2 = 7/4$$

6. (10 points) Use generating functions to find how many ways there are to put n identical balls into four boxes, in such a way that the first box has no more than 3 balls, the second has a multiple of 4 balls, the third has at least 5 balls, and there's no restriction on the number of balls in the fourth box.

使用生成函數找出將 n 個相同的球放入四個盒子中有多少種方法，使得第一個盒子不超過 3 個球，第二個盒子有 4 個球的倍數，第三個盒子至少有 5 個球，並且第四個盒子內的球數沒有限制。

Answer: $h_n = \binom{n-3}{2}$ **if** $n \geq 5$**, and** $h_n = 0$ **if** $n < 5$.

Solution：

The generating function is

$$\sum_{n=0}^{\infty} h_n x^n = (1 + x + x^2 + x^3)(1 + x^4 + x^8 + ...)(x^5 + x^6 + x^7 + ...)(1 + x + x^2 + ...)$$

$$= \frac{1-x^4}{1-x} \times \frac{1}{1-x^4} \times \frac{x^5}{1-x} \times \frac{1}{1-x}$$

$$= \frac{x^5}{(1-x)^3}$$

$$= x^5 \sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n$$

$$= \sum_{n=0}^{\infty} \binom{n+2}{2} x^{n+5}$$

$$= \sum_{n=5}^{\infty} \binom{n-3}{2} x^n$$

7. (15 points) Let h_n denote the number of ways to color the squares of a 1-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even, the number of squares colored white is odd and the number of squares colored blue is even. Determine the exponential generating function $g^{(e)}(x)$ for the sequence $h_0, h_1, h_2, ...$ and then find a simple formula for h_n.

令 h_n 表示用紅色、白色、藍色和綠色為 $1 \times n$ 方塊板著色方法的數量，並且紅色方塊的數量為偶數，白色方塊的數量是奇數，藍色方塊的數量是偶數。確定序列 $h_0, h_1, h_2, ...$ 的指數生成函數 $g^{(e)}(x)$ ，並以此找到 h_n 的簡單公式。

Answer: (a) $g^{(e)}(x) = \dfrac{e^{4x} + e^{2x} - e^{-2x} - 1}{2^3}$,

(b) $h_n = \dfrac{4^n + 2^n - (-2)^n}{2^3}$**, if** $n > 0$ **, and** $h_0 = 0$.

Solution：

The generating function is

$$
\begin{aligned}
\sum_{n=0}^{\infty} h_n \frac{x^n}{n!} &= (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...)(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + ...)(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...)(1 + \frac{x}{1!} + \frac{x^2}{2!} + ...) \\
&= \frac{e^x + e^{-x}}{2} \times \frac{e^x - e^{-x}}{2} \times \frac{e^x + e^{-x}}{2} \times e^x \\
&= \frac{e^{4x} + e^{2x} - e^{-2x} - 1}{2^3} \\
&= \sum_{n=1}^{\infty} \frac{4^n + 2^n - (-2)^n}{2^3} \frac{x^n}{n!}
\end{aligned}
$$

8. (10 points) Find the (ordinary) generating function for the infinite sequence $h_0, h_1, h_2, ...$ defined by $h_n = \binom{n}{2}$.

Answer: $\dfrac{x^2}{(1-x)^3}$

Solution：

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ...$$

$$\begin{aligned}\frac{x}{(1-x)^2} &= x\frac{d}{dx}\left(\frac{1}{1-x}\right)\\ &= x\frac{d}{dx}\left(1 + x + x^2 + x^3 + ... + x^n + ...\right)\\ &= x + 2x^2 + 3x^3 + ... + nx^n + ...\end{aligned}$$

$$\begin{aligned}\frac{x(x+1)}{(1-x)^3} &= x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)\\ &= x\frac{d}{dx}\left(x + 2x^2 + 3x^3 + ... + nx^n + ...\right)\\ &= x + 2^2x^2 + 3^2x^3 + ... + n^2x^n + ...\end{aligned}$$

$$\begin{aligned}\frac{x^2}{(1-x)^3} &= \frac{1}{2}\left(\frac{x(x+1)}{(1-x)^3} - \frac{x}{(1-x)^2}\right)\\ &= \frac{1}{2}\left(\left(x + 2^2x^2 + 3^2x^3 + ... + n^2x^n + ...\right) - \left(x + 2x^2 + 3x^3 + ... + nx^n + ...\right)\right)\\ &= 1x^2 + 3x^3 + ... + \frac{n^2-n}{2}x^n + ...\\ &= \sum_{n\leq 2}\binom{n}{2}x^n\end{aligned}$$

9. (10 points) Use combinatorial reasoning (組合解釋) to prove the identity

$$\binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \binom{n}{2}D_{n-2} + \binom{n}{3}D_{n-3} + ... + \binom{n}{n}D_0 = n!$$

with $D_0 = 1$ and D_k is the number of derangements of $\{1, 2, ..., k\}$.

Solution：

我上課有證過，而且是 ch6 作業 16。

10. (10 points) Find the determinant of the following $n \times n$ tri-diagonal (三對角線) matrix.

$$\begin{bmatrix} 5 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 5 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 5 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 5 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 5 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 2 & 5 \end{bmatrix}$$

Answer: $\frac{-1}{3} + \frac{4}{3}4^n$.

Solution：

Let t_n is the determinant of the above matrix.

It is easy to have $t_1 = |5| = 5$, $t_2 = \begin{vmatrix} 5 & 2 \\ 2 & 5 \end{vmatrix} = 25 - 4 = 21$.

$$\begin{vmatrix} 5 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 5 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 5 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 5 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 5 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 2 & 5 \end{vmatrix}_{n\times n} = 5\begin{vmatrix} 5 & 2 & 0 & \cdots & 0 & 0 \\ 2 & 5 & 2 & \cdots & 0 & 0 \\ 0 & 2 & 5 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 5 & 2 \\ 0 & 0 & 0 & \cdots & 2 & 5 \end{vmatrix}_{(n-1)\times(n-1)} - 2\begin{vmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 5 & 2 & \cdots & 0 & 0 \\ 0 & 2 & 5 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 5 & 2 \\ 0 & 0 & 0 & \cdots & 2 & 5 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= 5\begin{vmatrix} 5 & 2 & 0 & \cdots & 0 & 0 \\ 2 & 5 & 2 & \cdots & 0 & 0 \\ 0 & 2 & 5 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 5 & 2 \\ 0 & 0 & 0 & \cdots & 2 & 5 \end{vmatrix}_{(n-1)\times(n-1)} - 2\times 2\begin{vmatrix} 5 & 2 & \cdots & 0 & 0 \\ 2 & 5 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 5 & 2 \\ 0 & 0 & \cdots & 2 & 5 \end{vmatrix}_{(n-2)\times(n-2)}$$

Thus, we have $t_n = 5t_{n-1} - 4t_{n-2}$ with $t_1 = 5$, $t_2 = 21$. Therefore, $t_n = \frac{-1}{3} + \frac{4}{3}4^n$

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	15	10	10	10	10	10	15	10	10	10	110
Score:											