數學二離散數學 2022 秋, 期末考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 13 頁 (包含封面)，有 15 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。　**敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。　**遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) The sequence of numbers $h_0, h_1, ..., h_n, ...$ satisfies a recurrence relation whose characteristic roots are 2,2,2. What is the recurrence relation?

   Answer: recurrence relation: $h_n = 6h_{n-1} - 12h_{n-2} + 8h_{n-3}$.

   Solution：

   So the char. equation is $(x-2)^3 = x^3 - 6x^2 + 12x - 8$ and hence the recurrence relation is $h_n = 6h_{n-1} - 12h_{n-2} + 8h_{n-3}$

2. (10 points) Consider a I-by-n chessboard. Suppose we color each square of the chessboard with one of the two colors red and blue. Let h_n be the number of colorings in which no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then derive a formula for h_n .

   考慮 $1 \times n$ 棋盤。假設我們將棋盤中的每一個方格用紅或藍兩種顏色中的某種顏色著色。設 h_n 是使得沒有兩個紅色方格相鄰的著色方法數。求出並驗證h_n 的遞迴關係式，然後找出 h_n 的公式。

   Answer: recurrence relation: $h_n = h_{n-1} + h_{n-2}$, $h_0 = 1$ **and** $h_1 = 2$.

   The formula for h_n: $h_n = f_{n+2}$.

   Solution：

   By construction $h_0 = 1$ and $h_1 = 2$. We now find h_n for $n \geq 2$. Consider a coloring of the $1 \times n$ chessboard. The first square is colored red or blue. If it is blue, then there are h_{n-1} ways to color the remaining $n-1$ squares. If it is red, then the second square is blue, and there are h_{n-2} ways to color the remaining n $-$ 2 squares. Therefore $h_n = h_{n-1} + h_{n-2}$. Comparing the above data with the Fibonacci sequence we find $h_n = f_{n+2}$.

3. (10 points) The number of partitions of a set of n elements into k distinguishable(可區分的) boxes (some of which may be empty) is k^n. By counting in a different way, prove that

$$k^n = \binom{k}{1}1!S(n,1) + \binom{k}{2}2!S(n,2) + \ldots + \binom{k}{n}n!S(n,n)$$

(If $k > n$, define $S(n,k)$ to be 0.)

Solution：

Check Ch8 Theorem 8.2.5 and theorem 8.2.6.

4. (10 points) Find the Ferrers diagram of the given partition $\lambda : 30 = 8+6+6+4+3+2+1$, and the determine the conjugate partition λ^*.

Answer: $\lambda^* =$ **30=7+6+5+4+3+3+1+1**

5. (10 points) Let n be a positive integer. Let p_n^o be the number of partitions of n into odd parts, and let p_n^d be the number of partitions of n into distinct parts. In textbook, we establish a one-to-one correspondence between the two types of partitions. Then $p_n^o = p_n^d$. Please find the following corresponding partitions.

(a) the partition $\lambda_1 = 3^9 7^{11} 13^4 17^{20}$ will corresponding to λ_2.

$\lambda_2 =$ $3 \times 8 + 3 + 7 \times 8 + 7 \times 2 + 7 + 13 \times 4 + 17 \times 16 + 17 \times 4$

$= 24 + 3 + 56 + 14 + 7 + 52 + 272 + 68$.

(b) the partition $\tau_1 :\ 62 = 1 + 3 + 8 + 24 + 26$ will corresponding to τ_2.

$\tau_2 =$ $1^9 3^9 13^2$.

6. (10 points) (a) Write down a combinatorial model (組合模型) for the Catalan number. (請跟 (b) 小題不一樣的)

Solution：

Check ch 8-1

(b) Let $2n$ (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the n^{th} Catalan number C_n.

設在圓上選擇 2n 個（等間隔的）點。證明將這些點成對連接起來得到 n 條不相交線段的方法數等於第 n 個 Catalan 數 C_n。

Solution：

Ch 8 problem 1.

7. (10 points) (a) Write down the define for the Stirling number of the first $s(n,k)$ and it's recurrence relation.

Solution：

Check ch 8-2

(b) Write down the define for the Stirling number of second kind $S(n,k)$ and it's recurrence relation.

Solution：

Check ch 8-2

8. (10 points) Give the difference table for $h_n = 2n^3 - 3$. Using the difference table, find a closed formula for $\sum_{n=1}^{m} h_n$. (不需化簡)

Answer: $\sum_{n=1}^{m} h_n =$ $-3\binom{n+1}{1} + 2\binom{n+1}{2} + 12\binom{n+1}{3} + 12\binom{n+1}{4}$

Solution：

-3		-1		13		51		125
	2		14		38		74	
		12		24		36		
			12		12			
				0				

$h_n = -3\binom{n}{0} + 2\binom{n}{1} + 12\binom{n}{2} + 12\binom{n}{3}$

9. (10 points) The general term h_n of a sequence is a polynomial in n of degree 3. If the first four entries of the 0^{th} row of its difference table are $2, 3, -2, 7$, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.

Answer: $h_5 =$ $2\binom{5}{0} + 1\binom{5}{1} - 6\binom{5}{2} + 20\binom{5}{3} = 147$. $h_n =$ $2\binom{n}{0} + 1\binom{n}{1} - 6\binom{n}{2} + 20\binom{n}{3}$.

$\sum_{k=0}^{n} h_k =$ $2\binom{n+1}{1} + 1\binom{n+1}{2} - 6\binom{n+1}{3} + 20\binom{n+1}{4}$.

Solution：

2		3		-2		7	...	
	1		-5		9	...		
		-6		14	...			
			20	...	...			
				0	...			

$h_n = 2\binom{n}{0} + 1\binom{n}{1} - 6\binom{n}{2} + 20\binom{n}{3}$

10. (10 points) Solve the recurrence relation $h_n = 6h_{n-1} - 9h_{n-2}$ with initial values $h_0 = 2$ and $h_1 = 7$.

Answer: $h_n = 2 \times 3^n + n \times 3^{n-1}$.

Solution：

Rewrite the given recurrence relation as $h_n - 6h_{n-1} + 9h_{n-2} = 0$ The characteristic equation of the recurrence relation is $x^2 - 6x + 9 = (x-3)^2 = 0$. Thus we have the characteristic are $x = 3$.

Therefore,

$$h_n = c_1 \times 3^n + c_2 \times n \times 3^n$$

Using the initial condition

$$h_0 = 2 = c_1 \times 3^0 + c_2 \times 0 \times 3^0$$
$$h_1 = 7 = c_1 \times 3^1 + c_2 \times 1 \times 3^1$$

.

We have

$$c_1 = 2,\ c_2 = \frac{1}{3}$$

11. (10 points) Solve the nonhomogeneous recurrence relation $h_n = 4h_{n-1} + 3 \times 2^n$ with initial values $h_0 = 1$.

Answer: $4^{n+1} - 3 \cdot 2^n$.

12. (10 points) Use generating functions to find how many ways there are to put n identical balls into four boxes, in such a way that the first box has no more than 4 balls, the second has a multiple of 5 balls, the third has at least 5 balls and has odd number of balls, and there's at most one balls in the fourth box.

使用生成函數找出將 n 個相同的球放入四個盒子中有多少種方法，使得第一個盒子最多 4 個球，第二個盒子有 5 個球的倍數，第三個盒子至少有 5 個球而且是奇數顆球，並且第四個盒子內的最多一顆球。

Answer: $n-4$.

Solution：

The generating function is

$$\begin{aligned} g(x) &= (1+x+x^2+x^3+x^4)(1+x^5+x^{10}+...)(x^5+x^7+x^9+...)(1+x) \\ &= \sum_{n\geq 5}(n-4)x^n \end{aligned}$$

13. (10 points) Let h_n denote the number of ways to color the squares of a 1-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even, the number of squares colored white is odd. Determine the exponential generating function $g^{(e)}(x)$ for the sequence $h_0, h_1, h_2, ...$ and then find a simple formula for h_n.

令 h_n 表示用紅色、白色、藍色和綠色為 $1 \times n$ 方塊板著色方法的數量，並且紅色方塊的數量為偶數，白色方塊的數量是奇數。確定序列 $h_0, h_1, h_2, ...$ 的指數生成函數 $g^{(e)}(x)$ ，並以此找到 h_n 的簡單公式。

Answer: (a) $g^{(e)}(x) =$ $\frac{1}{4}(e^{4x} - 1)$,

(b) $h_n =$ 4^{n-1}**, if** $n \geq 1$ **and** $h_0 = 0$.

Solution：

The generating function is

$$
\begin{aligned}
g(x) &= (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!})(1 + \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + ...)(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...)^2 \\
&= \frac{1}{4}(e^{4x} - 1) \\
&= \sum_{n \geq 1} \frac{1}{4} \frac{(4x)^n}{n!}
\end{aligned}
$$

14. (10 points) Find the determinant of the following $n \times n$ tri-diagonal (三對角線) matrix.

$$\begin{bmatrix} 10 & 3 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 10 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 10 & 3 & \cdots & 0 & 0 \\ 0 & 0 & 3 & 10 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 10 & 3 \\ 0 & 0 & 0 & 0 & \cdots & 3 & 10 \end{bmatrix}$$

Answer: $\frac{-1}{8} + \frac{9}{8}9^n$.

Solution：

Let t_n is the determinant of the above matrix.

It is easy to have $t_1 = |10| = 10$, $t_2 = \begin{vmatrix} 10 & 3 \\ 3 & 10 \end{vmatrix} = 100 - 9 = 91$.

$$\begin{vmatrix} 10 & 3 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 10 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 10 & 3 & \cdots & 0 & 0 \\ 0 & 0 & 3 & 10 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 10 & 3 \\ 0 & 0 & 0 & 0 & \cdots & 3 & 10 \end{vmatrix}_{n\times n} = 10 \begin{vmatrix} 10 & 3 & 0 & \cdots & 0 & 0 \\ 3 & 10 & 3 & \cdots & 0 & 0 \\ 0 & 3 & 10 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 10 & 3 \\ 0 & 0 & 0 & \cdots & 3 & 10 \end{vmatrix}_{(n-1)\times(n-1)} - 3 \times 3 \begin{vmatrix} 10 & 3 & \cdots & 0 & 0 \\ 3 & 10 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 10 & 3 \\ 0 & 0 & \cdots & 3 & 10 \end{vmatrix}_{(n-3)\times(n-3)}$$

Thus, we have $t_n = 10t_{n-1} - 9t_{n-2}$ with $t_1 = 10$, $t_2 = 91$. Therefore, $t_n = \frac{-1}{8} + \frac{9}{8}9^n$

15. (10 points) Let f_n is the n^{th} Fibonacci number. Prove that f_n is even if and only if n is divisible by 3.

Solution：
check Ch 7 problem 3

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8
Points:	10	10	10	10	10	10	10	10
Score:								

Question:	9	10	11	12	13	14	15	Total
Points:	10	10	10	10	10	10	10	150
Score:								