應數二離散數學 2023 春, 期末考 解答

學號: ________________________________, 姓名: ________________________________

本次考試共有 13 頁 (包含封面)，有 13 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Find the (ordinary) generating function for the infinite sequence $h_0, h_1, h_2, \ldots.$ defined by $h_n = n(n+1)$.

   Answer: $\dfrac{2x^2}{(1-x)^3}$

   Solution：

   From Ch 7.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

$$\begin{aligned}\frac{x}{(1-x)^2} &= x\frac{d}{dx}\left(\frac{1}{1-x}\right)\\ &= x\frac{d}{dx}\left(1 + x + x^2 + x^3 + \ldots + x^n + \ldots\right)\\ &= x + 2x^2 + 3x^3 + \ldots + nx^n + \ldots\end{aligned}$$

$$\begin{aligned}\frac{x(x+1)}{(1-x)^3} &= x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)\\ &= x\frac{d}{dx}\left(x + 2x^2 + 3x^3 + \ldots + nx^n + \ldots\right)\\ &= x + 2^2x^2 + 3^2x^3 + \ldots + n^2x^n + \ldots\end{aligned}$$

$$\begin{aligned}\frac{2x^2}{(1-x)^3} &= \left(\frac{x(x+1)}{(1-x)^3} + \frac{x}{(1-x)^2}\right)\\ &= \left(\left(x + 2^2x^2 + 3^2x^3 + \ldots + n^2x^n + \ldots\right) + \left(x + 2x^2 + 3x^3 + \ldots + nx^n + \ldots\right)\right)\\ &= \sum_{n\geq 0}(n^2+n)x^n\end{aligned}$$

2. (10 points) Find the sum and product of two generating function.

   (a) Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $B(x) = \sum_{n=0}^{\infty} b_n x^n$, $C(x) = \sum_{n=0}^{\infty} c_n x^n$, $D(x) = \sum_{n=0}^{\infty} d_n x^n$. If $C(x) = A(x) + B(x)$ and $D(x) = A(x)B(x)$, find c_n and d_n.

   (b) Let $E(x) = \sum_{n=0}^{\infty} e_n \frac{x^n}{n!}$, $F^{(e)}(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$, $G(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}$, $H(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$. If $G(x) = E(x) + F(x)$ and $H(x) = E(x)F(x)$, find g_n and h_n.

   Answer: $c_n =$ $a_n + b_n$, $d_n =$ $\sum_{k\geq 0}^{n} a_{n-k} b_k$, $g_n =$ $e_n + f_n$, $h_n =$ $\sum_{k\geq 0}^{n} \binom{n}{k} e_{n-k} f_k$
   .

3. (10 points) Give the difference table for $h_n = 2n^3 - 3$. Using the difference table, find a closed formula for $\sum_{n=1}^{m} h_n$. (不需化簡)

   Answer: $\sum_{n=1}^{m} h_n =$ $-3\binom{m+1}{1} + 2\binom{m+1}{2} + 12\binom{m+1}{3} + 12\binom{m+1}{4}$

   Solution：

   -3 -1 13 51 125

   2 14 38 74

   12 24 36

   12 12

   0

   $h_n = -3\binom{n}{0} + 2\binom{n}{1} + 12\binom{n}{2} + 12\binom{n}{3}$

4. (10 points) Let n be a positive integer. Let p_n^o be the number of partitions of n into odd parts, and let p_n^d be the number of partitions of n into distinct parts. In textbook, we establish a one-to-one correspondence between the two types of partitions. Then $p_n^o = p_n^d$. Please find the following corresponding partitions.

(a) the partition $\lambda_1 = 3^9 9^{11} 13^4 19^{20}$ will corresponding to λ_2.

$\lambda_2 =$ $\underline{3 \times 8 + 3 + 9 \times 8 + 9 \times 2 + 9 + 13 \times 4 + 19 \times 16 + 19 \times 4}$

$\underline{= 24 + 3 + 72 + 18 + 9 + 52 + 304 + 76}$.

(b) the partition $\tau_1 :\ 67 = 1 + 3 + 8 + 9 + 14 + 24$ will corresponding to τ_2.

$\tau_2 =$ $\underline{1^9 3^9 7^2 9^1}$.

5. (10 points) The general term h_n of a sequence is a polynomial in n. If the first four entries of the 0^{th} row of its difference table are $0, 16, 50, 108, 196, \underline{320}, 486, \ldots$, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.(不需化簡)

Answer: $h_5 =$ $\underline{0\binom{5}{0} + 16\binom{5}{1} + 18\binom{5}{2} + 6\binom{5}{3} = 320}$. $h_n =$ $\underline{0\binom{n}{0} + 16\binom{n}{1} + 18\binom{n}{2} + 6\binom{n}{3}}$.

$\sum_{k=0}^{n} h_k =$ $\underline{0\binom{n+1}{1} + 16\binom{n+1}{2} + 18\binom{n+1}{3} + 6\binom{n+1}{4}}$.

Solution：

0	16	50	108	196	320	...
16	34	58	88	224	...	
18	24	30	36	...		
6	6	6	...			
0	0	...				

$h_n = 0\binom{n}{0} + 16\binom{n}{1} + 18\binom{n}{2} + 6\binom{n}{3}$

6. (10 points) Solve the nonhomogeneous recurrence relation $h_n = 2h_{n-1} + 5^n$ with initial values $h_0 = 3$.

   Answer: $h_n =$ $(5^{n+1} + 2^{n+2})/3$.

   Solution：

$$\begin{aligned}(1-2x)g(x) =& 3 + (\frac{1}{1-5x} - 1) = \frac{3-10x}{1-5x}\\ g(x) =& \frac{3-10x}{(1-5x)(1-2x)} = \frac{\frac{5}{3}}{1-5x} + \frac{\frac{4}{3}}{1-2x}\\ =& \frac{5}{3}\sum_{n\geq 0}(5x)^n + \frac{4}{3}\sum_{n\geq 0}(2x)^n\end{aligned}$$

7. (10 points) Use generating functions to determine the number of integral solutions of the equation

$$x_1 + 4x_2 + x_3 + x_4 = n,$$

that satisfy

$$0 \le x_1 \le 3, 0 \le x_2,\ 5 \le x_3,\ 0 \le x_4$$

Answer: $h_n = \binom{n-3}{2}$ **if** $n \ge 5$**, and** $h_n = 0$ **if** $n < 5$.

Solution：

The generating function is

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n x^n &= (1+x+x^2+x^3)(1+x^4+x^8+...)(x^5+x^6+x^7+...)(1+x+x^2+...) \\
&= \frac{1-x^4}{1-x} \times \frac{1}{1-x^4} \times \frac{x^5}{1-x} \times \frac{1}{1-x} \\
&= \frac{x^5}{(1-x)^3} \\
&= x^5 \sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n \\
&= \sum_{n=0}^{\infty} \binom{n+2}{2} x^{n+5} \\
&= \sum_{n=5}^{\infty} \binom{n-3}{2} x^n
\end{aligned}$$

8. (10 points) Let h_n denote the number of ways to color the squares of a 1-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even, the number of squares colored white and blue is odd. Determine the exponential generating function $g^{(e)}(x)$ for the sequence $h_0, h_1, h_2, ...$ and then find a simple formula for h_n.

令 h_n 表示用紅色、白色、藍色和綠色為 $1 \times n$ 方塊板著色方法的數量，並且紅色方塊的數量為偶數，白色和藍色方塊的數量都是奇數。確定序列 $h_0, h_1, h_2, ...$ 的指數生成函數 $g^{(e)}(x)$，並以此找到 h_n 的簡單公式。

Answer: (a) $g^{(e)}(x) =$ $\frac{1}{8}(e^{4x} - e^{2x} + e^{-2x} - 1)$,

(b) $h_n =$ $\frac{1}{8}(4^n - 2^n + (-2)^n)$**, if** $n \geq 1$ **and** $h_0 = 0$.

Solution：

The generating function is

$$
\begin{aligned}
g(x) &= (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!})(1 + \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + ...)^2(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...) \\
&= \frac{(e^x + e^{-x})}{2}\frac{(e^x - e^{-x})}{2}\frac{(e^x - e^{-x})}{2}e^x \\
&= \frac{1}{8}(e^{4x} - e^{2x} + e^{-2x} - 1)
\end{aligned}
$$

9. (10 points) Find the determinant of the following $n \times n$ tri-diagonal (三對角線) matrix.

$$\begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 4 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix}$$

Answer: $\frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1})$.

Solution：

Let t_n is the determinant of the above matrix.

It is easy to have $t_1 = |4| = 4$, $t_2 = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 16 - 1 = 15$.

$$\begin{vmatrix} 4 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 4 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 \end{vmatrix}_{n\times n} = 4 \begin{vmatrix} 4 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 4 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{vmatrix}_{(n-1)\times(n-1)} - 1 \times 1 \begin{vmatrix} 4 & 1 & \cdots & 0 & 0 \\ 1 & 4 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 4 & 1 \\ 0 & 0 & \cdots & 1 & 4 \end{vmatrix}_{(n-1)\times(n-1)}$$

Thus, we have $t_n = 4t_{n-1} - t_{n-2}$ with $t_1 = 4$, $t_2 = 15$. Since $t_2 = 15 = 4t_1 - t_0 = 4 \times 4 - t_0$, we have $t_0 = 1$.

$$x^2 = 4x - 1 \Rightarrow x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

Therefore, $t_n = c_1(2+\sqrt{3})^n + c_2(2-\sqrt{3})^n$ with $t_0 = 1, t_1 = 4$.

$$\begin{cases} 1 = t_0 & = c_1 + c_2 \\ 4 = t_1 & = c_1(2+\sqrt{3}) + c_2(2-\sqrt{3}) \end{cases}$$

$$t_n = \frac{2+\sqrt{3}}{2\sqrt{3}}(2+\sqrt{3})^n + \frac{-2+\sqrt{3}}{2\sqrt{3}}(2-\sqrt{3})^n = \frac{1}{2\sqrt{3}}(2+\sqrt{3})^{n+1} - \frac{1}{2\sqrt{3}}(2-\sqrt{3})^{n+1}$$

10. (10 points) Let a_n equal the number of ternary strings of length n made up of 0's, 1's, and 2's, such that the substrings 00, 01, 10, and 11 never occur. Prove that

$$a_n = a_{n-1} + 2a_{n-2},\ (n \geq 2),$$

with $a_0 = 1$ and $a_1 = 3$. Then find a formula for a_n.

設 a_n 等於由 0、1 和 2 組成的長度為 n 的三元字符串的數量，並且這樣的子字符串中永遠不會出現 00、01、10 和 11。證明

$$a_n = a_{n-1} + 2a_{n-2},\ (n \geq 2),$$

初始值 $a_0 = 1$ 和 $a_1 = 3$. 並且找出 a_n 的表達式.

Answer: $a_n =$ $\frac{1}{3}(2^{n+2} + (-1)^{n+1})$.

Solution：

ch7 problem 40, 記得要證明 ! !

Rewrite the given recurrence relation as $a_n - a_{n-1} - 2a_{n-2} = 0$ The characteristic equation of the recurrence relation is $x^2 - x - 2 = (x-2)(x+1) = 0$. Thus we have the characteristic are $x = 3$.

Therefore,

$$h_n = c_1 \times 2^n + c_2 \times (-1)^n$$

Using the initial condition

$$h_0 = 1 = c_1 \times 2^0 + c_2 \times 0 \times (-1)^0$$
$$h_1 = 3 = c_1 \times 2^1 + c_2 \times 1 \times (-1)^1$$

.

We have

$$c_1 = \frac{4}{3},\ c_2 = \frac{-1}{3}$$

11. (10 points) Prove that $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ and $s(n, n-1) = \binom{n}{2}$, where $s(n, k)$ is the Stirling numbers of the first kind and $S(n, k)$ is the second kind.

Solution：

This is Ch 8 problem 12(d) and 19(b).

推薦組合證明，但是若用代數證明，可以使用遞迴歸式。

12. (10 points) Let f_n is the n^{th} Fibonacci number. Prove that f_n is divisible by 3 if and only if n is divisible by 4.

Solution：

check Ch 7 problem 3

13. (10 points) Let $2n$ (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the n^{th} Catalan number C_n.

設在圓上選擇 2n 個（等間隔的）點。證明將這些點成對連接起來得到 n 條不相交線段的方法數等於第 n 個 Catalan 數 C_n。

Solution：

Ch 8 problem 1.

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
Points:	10	10	10	10	10	10	10	10	10	10	10	10	10	130
Score:														