數學二離散數學 2023 秋, 期末考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 11 頁 (包含封面)，有 11 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Find the (ordinary) generating function for the infinite sequence $h_0, h_1, h_2, ...$ defined by $h_n = n(n-1)$.

   Answer: $\dfrac{2x^2}{(1-x)^3}$

   Solution：

   From Ch 7.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ...$$

$$\begin{aligned}\frac{x}{(1-x)^2} &= x\frac{d}{dx}\left(\frac{1}{1-x}\right)\\ &= x\frac{d}{dx}\left(1 + x + x^2 + x^3 + ... + x^n + ...\right)\\ &= x + 2x^2 + 3x^3 + ... + nx^n + ...\end{aligned}$$

$$\begin{aligned}\frac{x(x+1)}{(1-x)^3} &= x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)\\ &= x\frac{d}{dx}\left(x + 2x^2 + 3x^3 + ... + nx^n + ...\right)\\ &= x + 2^2x^2 + 3^2x^3 + ... + n^2x^n + ...\end{aligned}$$

$$\begin{aligned}\frac{2x^2}{(1-x)^3} &= \left(\frac{x(x+1)}{(1-x)^3} - \frac{x}{(1-x)^2}\right)\\ &= \left(\left(x + 2^2x^2 + 3^2x^3 + ... + n^2x^n + ...\right) - \left(x + 2x^2 + 3x^3 + ... + nx^n + ...\right)\right)\\ &= \sum_{n\geq 0}(n^2 - n)x^n\end{aligned}$$

2. (10 points) Determine the conjugate of each of the following partitions: 34 = 9+8+6+6+3+2

Answer: 34 = $6+6+5+4+4+4+2+2+1$

Solution：

		6	6	5	4	4	4	2	2	1
		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
9	...	o	o	o	o	o	o	o	o	o
8	...	o	o	o	o	o	o	o	o	
6	...	o	o	o	o	o	o			
6	...	o	o	o	o	o	o			
3	...	o	o	o						
2	...	o	o							

3. (10 points) Let p_n^s equal the number of self-conjugate partitions of n. Find p_{15}^s. *Hint:* By Theorem 8.3.2, let p_n^t be the number of partitions of n into distinct odd parts. Then $p_n^s = p_n^t$.

Answer: $p_{15}^s =$ 4

Solution：

p_{15}^t	p_{15}^s
15	8+1+1+1+1+1+1+1
11+3+1	6+3+3+1+1+1
9+5+1	5+4+3+2+1
7+5+3	4+4+4+3

4. (10 points) Let n be a positive integer. Let P_n^o be the set of partitions of n into odd parts, and let P_n^d be the set of partitions of n into distinct parts. In textbook, we establish a one-to-one correspondence between the two types of partitions. Then $|P_n^o| = |P_n^d|$. Please find the following corresponding partitions.

(a) the partition λ_1 : $453 = 5^{11}9^{9}11^{4}13^{21} \in P_n^o$ will corresponding to $\lambda_2 \in P_n^d$.

$\lambda_2 =$ <u>$5 + 9 + 10 + 13 + 40 + 44 + 52 + 72 + 208$</u> ..

(b) the partition τ_1 : $86 = 1 + 3 + 4 + 18 + 20 + 40 \in P_n^d$ will corresponding to $\tau_2 \in P_n^o$.

$\tau_2 =$ <u>$1^5 3^1 5^{12} 9^2$</u> .

Solution：

(a) $5\times(1+2+8)+9\times(1+8)+11\times(4)+13\times(1+4+16) = 5+10+40+9+72+44+13+52+208$

(b) $1+3+4+18+20+40 = 1+3+1\times4+9\times2+5\times4+5\times8 = 1\times(1+4)+3+5\times(4+8)+9\times2$

5. (10 points) The general term h_n of a sequence is a polynomial in n. If the first few elements are $3, 2, 7, 24, 59, 118, 207, \ldots$, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.(不需化簡)

Answer: $h_n =$ $3\binom{n}{0} - \binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$.

$\sum_{k=0}^{n} h_k =$ $3\binom{n+1}{1} - \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}$.

Solution：

3		2		7		24		59		118	...
	-1		5		17		35		59	...	
		6		12		18		24	...		
			6		6		6	...			
				0		0	...				

6. (15 points) Solve the nonhomogeneous recurrence relation $h_n = 6h_{n-1} - 9h_{n-2} + 5^n$ with initial values $h_0 = 3$, $h_1 = 12$. 提示：你可以分成 homogeneous 跟 non-homogeneous 的兩部分算。

Answer: $h_n =$ $\frac{-13}{4}3^n - \frac{19}{6}n3^n + \frac{25}{4}5^n$.

Solution：

Ch 7

non-homogeneous:

Let $h_n = c_1 5^n \;\Rightarrow\; c_1 = \frac{25}{4}$

homogeneous:

$h_n - 6h_{n-1} + 9h_{n-2} = 0$

$x^2 - 6x + 9 = (x-3)^2 \;\Rightarrow\; x = 3$ (重根)

$h_n = c_2 3^n + c_3 n 3^n$.

exact solution:

$h_n = c_2 3^n + c_3 n 3^n + \frac{25}{4}5^n$ 代回初始值

$$h_n = \frac{-13}{4}3^n - \frac{19}{6}n3^n + \frac{25}{4}5^n$$

7. (10 points) Determine the generating function for the number h_n of bags of fruit of apples, oranges, bananas, and pears in which there are at least five oranges, a multiple of four number of bananas, at most three pear and no rule for apple. Then find a formula for h_n from the generating function.

一袋水果包含蘋果、橙子、香蕉和梨，其中至少有五個橙子，香蕉的數量是四的倍數，梨的數量最多為三，而對蘋果沒有特定規則。令 h_n 為一袋 n 個水果的可能組合的數量，找出 h_n 的生成函數，並從而找到 h_n 的公式。

Answer: (a) $\frac{x^5}{(1-x)^3}$

(b) $h_n = \binom{n-3}{2}$ **if** $n \geq 5$**, and** $h_n = 0$ **if** $n < 5$.

Solution：

The generating function is

$$\begin{aligned}\sum_{n=0}^{\infty} h_n x^n &= (1+x+x^2+x^3)(1+x^4+x^8+...)(x^5+x^6+x^7+...)(1+x+x^2+...)\\ &= \frac{1-x^4}{1-x} \times \frac{1}{1-x^4} \times \frac{x^5}{1-x} \times \frac{1}{1-x}\\ &= \frac{x^5}{(1-x)^3}\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} h_n x^n &= \frac{x^5}{(1-x)^3}\\ &= x^5 \sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n\\ &= \sum_{n=0}^{\infty} \binom{n+2}{2} x^{n+5}\\ &= \sum_{n=5}^{\infty} \binom{n-3}{2} x^n\end{aligned}$$

8. (10 points) Let h_n denote the number of n-digit numbers with all digits at least 4, such that 4 and 6 each occur an even number of times, and 5 and 7 each occur at least once, there being no restriction on the digits 8 and 9. Determine the exponential generating function $g^{(e)}(x)$ for the sequence $h_0, h_1, h_2, ...$ and then find a simple formula for h_n.

令 h_n 表示確定所有位數至少為 4 的 n 位數的數量，其中 4 和 6 都出現偶數次，且 5 和 7 至少各出現一次，對於數字 8 和 9 沒有任何限制。確定序列 $h_0, h_1, h_2, ...$ 的指數生成函數 $g^{(e)}(x)$ ，並以此找到 h_n 的簡單公式。

Answer: (a) $g^{(e)}(x) =$ $\frac{1}{4}(e^{6x} - 2e^{5x} + 3e^{4x} - 4e^{3x} + 3e^{2x} - 2e^{x} + 1)$,

(b) $h_n =$ $\frac{1}{4}(6^n - 2 \times 5^n + 3 \times 4^n - 4 \times 3^n + 3 \times 2^n - 2)$**, if** $n \geq 1$ **and** $h_0 = 0$.

Solution：

The generating function is

$$
\begin{aligned}
g(x) &= (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!})^2(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...)^2(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...)^2 \\
&= (\frac{(e^x + e^{-x})}{2})^2(e^x - 1)^2(e^x)^2 \\
&= \frac{1}{4}(e^{2x} + 2 + e^{-2x})(e^{2x} - 2e^x + 1)e^{2x} \\
&= \frac{1}{4}(e^{6x} - 2e^{5x} + 3e^{4x} - 4e^{3x} + 3e^{2x} - 2e^x + 1)
\end{aligned}
$$

9. (10 points) Find the determinant of the following $n \times n$ tri-diagonal (三對角線) matrix.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

Answer: ____ .

Solution：

Let t_n is the determinant of the above matrix.

It is easy to have $t_1 = |1| = 4$, $t_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 - (-1) = 2$.

$$\begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{vmatrix}_{n\times n} = 1 \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{vmatrix}_{(n-1)\times(n-1)} - 1 \times (-1) \begin{vmatrix} 1 & -1 & \cdots & 0 & 0 \\ 1 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & -1 \\ 0 & 0 & \cdots & 1 & 1 \end{vmatrix}_{(n-1)\times(n}$$

Thus, we have $t_n = t_{n-1} + t_{n-2}$ with $t_1 = 1$, $t_2 = 2$. It is easy to see that the $t_n = f_{n-1}$ is the Fibonacci number.

10. (10 points) The number of partitions of a set of n elements into k distinguishable(可區分的) boxes (some of which may be empty) is k^n. By counting in a different way, prove that

$$k^n = \binom{k}{1}1!S(n,1) + \binom{k}{2}2!S(n,2) + ... + \binom{k}{n}n!S(n,n)$$

(If $k > n$, define $S(n,k)$ to be 0.)

Solution：

Check Ch8 Theorem 8.2.5 and theorem 8.2.6.

11. (10 points) Let m and n be nonnegative integers with $n \geq m$. There are m + n people in line to get into a theater for which admission is 50 cents. Of the m + n people, n have a 50-cent piece and m have a $1 dollar bill. The box office opens with an empty cash register. Show that the number of ways the people can line up so that change is available when needed is

$$\frac{n-m+1}{n+1}\binom{m+n}{m}$$

讓 m 和 n 為非負整數，且滿足 $n \geq m$。有 m + n 人排隊進入一個門票為 50 美分的劇院。在這 m + n 人中，有 n 人支付一個 50 美分的硬幣，而 m 人支付一張 1 美元的鈔票。售票亭在現金櫃檯是空的情況下開放。證明這些人排隊的方式中，確保需要時能提供找零的方式數為上式。

Solution：

Ch 8 problem 5.

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	10	10	10	10	15	10	10	10	10	10	115
Score:												