應數二離散數學 2024 春, 第一次期中考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 8 頁 (包含封面)，有 11 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Find integers a and b such that

$$m^2 = a\binom{m}{2} + b\binom{m}{1}$$

for all m. Then use it to find the sum the series $1^2 + 2^2 + ... + n^2$.

Answer: $a =$ **2** , $b =$ **1** , $1^2 + 2^2 + ... + n^2 =$ $2\binom{n+1}{3} + \binom{n+1}{2} = \frac{n(n+1)(2n+1)}{6}$.

2. (10 points) Assume there is a standard deck of 52 cards. a) How many cards must be selected to guarantee that at least three cards of the same suit are chosen? b) How many cards must be selected to guarantee that at least three hearts are selected

假設有一副標準的 52 張牌。a) 為了確保至少選擇到三張相同花色的牌，必須選擇多少張牌？b) 為了確保至少選擇到三張紅心的牌，必須選擇多少張牌？

Answer: (a) **4*2+1=9** , (b) **13*3+2+1=42** .

Solution：

一副標準的 52 張撲克牌，組成為四個花色（黑桃、紅心、方塊、梅花），每種花色有 13 個不同的點數。

(a) 沒有三張相同的花色 $\Rightarrow$ 四種花色，每種最多兩張 $= 2 \times 4 = 8$。

(b) 沒有三張紅心 $\Rightarrow$ 紅心最多兩張，其他三種花色最多各 13 張。

3. (10 points) In a standard deck of 52 playing cards, (a) how many different types of full houses are there (consisting of three cards of the same rank and two cards of another rank), and (b) how many different types of straights are there (consisting of five consecutive cards in any suit, ranging from A, 2, 3, 4, 5 to 9, 10, J, Q, K)? 在一副標準的 52 張撲克牌中，(a) 有多少種不同的葫蘆（由三張相同點數的牌和兩張另一點數的牌組成），以及 (b) 有多少種不同的順子（由任意花色的五張連續點數的牌組成，從 A、2、3、4、5 到 9、10、J、Q、K）？

Answer: (a) $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$, (b) $9 \times (4^5 - 4) = 9180$.

Solution：

(a) 先選『三張相同點數的牌』的點數，再選其花色 $\binom{13}{1}\binom{4}{3}$。再從未曾選中得點數中，選取『兩張相同點數得牌』的點數，再選其花色 $\binom{12}{1}\binom{4}{2}$。

(b)(i) 『五張連續點數的牌』，組合有 [A,2,3,4,5], [2,3,4,5,6], [3,4,5,6,7], [4,5,6,7,8], [5,6,7,8,9], [6,7,8,9,10], [7,8,9,10,J], [8,9,10,J,Q], [9,10,J,Q,K] 共 9 種。花色的可能性為 4^5。

(b)(ii) 若是嚴格只問順子，則不含同花順（五張連續點數且同花色的牌）所以是共 9 種點數組合，乘上，花色的可能性為 $4^5 - 4$。

這次因為題目沒特別表明，所以 (b)(i) 也算對！

4. (10 points) In which position does the subset 2579 occur in the lexicographic order of the 4-subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

子集 2579 出現在集合 $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ 的 4-子集的字典序的哪個位置上？

Answer: **86**.

Solution：

By textbook Theroem 4.4.2，$n = 9, r = 4, a_1 = 2, a_2 = 5, a_3 = 7, a_4 = 9$。

$$
\begin{aligned}
&\binom{9}{4} - \binom{9-2}{4} - \binom{9-5}{3} - \binom{9-7}{2} - \binom{9-9}{1} \\
=&\binom{9}{4} - \binom{7}{4} - \binom{4}{3} - \binom{2}{2} - \binom{0}{1} \\
=&\frac{9!}{4!5!} - \frac{7!}{4!3!} - 4 - 1 - 0 \\
=&86
\end{aligned}
$$

5. (10 points) A student has 35 days to prepare for an examination. From past experience she knows that she will require no more than 53 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day, however), there is a succession of days during which she will have studied exactly 16 hours.

   一個學生有 35 天用來準備考試。根據以往的經驗，她知道她的唸書時間不超過 53 小時，她也希望她每天至少要唸一個小時的書。證明：無論她怎麼安排她的讀書計畫（每天的唸書時間都是一個整數小時），都存在連續的某幾天，在這幾天裡，她恰好唸了 16 個小時的書。

   Solution：

   Similar to textbook page 71, section 3.1 application 4.

6. (10 points) Determine the mobile integers in

$$\overrightarrow{3}\ \overleftarrow{7}\ \overleftarrow{1}\ \overrightarrow{6}\ \overleftarrow{5}\ \overleftarrow{2}\ \overrightarrow{4}$$

   and then use the algorithm for generating the next six permutations.

   Answer: the mobile integers are **7,6** .

   (1) $\overleftarrow{7}\ \overrightarrow{3}\ \overleftarrow{1}\ \overrightarrow{6}\ \overleftarrow{5}\ \overleftarrow{2}\ \overrightarrow{4}$. (2) $\overrightarrow{7}\ \overrightarrow{3}\ \overleftarrow{1}\ \overleftarrow{5}\ \overrightarrow{6}\ \overleftarrow{2}\ \overrightarrow{4}$.

   (3) $\overrightarrow{3}\ \overrightarrow{7}\ \overleftarrow{1}\ \overleftarrow{5}\ \overrightarrow{6}\ \overleftarrow{2}\ \overrightarrow{4}$. (4) $\overrightarrow{3}\ \overleftarrow{1}\ \overrightarrow{7}\ \overleftarrow{5}\ \overrightarrow{6}\ \overleftarrow{2}\ \overrightarrow{4}$.

   (5) $\overrightarrow{3}\ \overleftarrow{1}\ \overleftarrow{5}\ \overrightarrow{7}\ \overrightarrow{6}\ \overleftarrow{2}\ \overrightarrow{4}$. (6) $\overrightarrow{3}\ \overleftarrow{1}\ \overleftarrow{5}\ \overrightarrow{6}\ \overrightarrow{7}\ \overleftarrow{2}\ \overrightarrow{4}$.

7. (10 points) Determine the number of integral solutions of the equation

$$x + y + z + w = 21$$

that satisfy

$$4 \leq x \leq 9, \ \ -6 \leq y \leq 1, \ 5 \leq z \leq 9, \ 5 \leq w \leq 9.$$

Answer: $\binom{16}{13} - [\binom{10}{7} + \binom{8}{5} + \binom{11}{8} + \binom{11}{8}] + [\binom{5}{2} + \binom{5}{2} + \binom{3}{0} + \binom{3}{0} + \binom{6}{3}] = 96$.

8. (10 points) Construct a permutation whose inversion sequences are 6, 4, 4, 3, 1, 1, 2, 0, 0.

   Answer: **856423197** .

9. (10 points) Find the remainder of $x^{100}-1$ divided by $(x+1)^3$.

   找出 $x^{100}-1$ 除以 $(x+1)^3$ 的餘式

   Answer: $\binom{100}{2}(x+1)^2-\binom{100}{1}(x+1)=4950x^2+9800x+4850$

Solution：

$$
\begin{aligned}
x^{100}-1 =& ((x+1)-1)^{100}-1 \\
=& \left(\sum_{k=0}^{100}\binom{100}{100-k}(-1)^{100-k}(x+1)^k\right)-1 \\
=& \left(\sum_{k=3}^{100}\binom{100}{100-k}(-1)^{100-k}(x+1)^k\right)+\left(\sum_{k=0}^{2}\binom{100}{100-k}(-1)^{100-k}(x+1)^k\right)-1 \\
=& (x+1)^3\times F(x)+\left((x+1)^0-\binom{100}{99}(x+1)+\binom{100}{2}(x+1)^2\right)-1 \\
=& (x+1)^3\times F(x)+1-100(x+1)+4950(x+1)^2-1 \\
=& (x+1)^3\times F(x)+\left(4950x^2+9800x+4850\right)
\end{aligned}
$$

10. (10 points) Show that if n + 1 distinct integers are chosen from the set {l, 2, ... , 3n}, then there are always two which differ by at most 2.

證明：如果從集合 1, 2, ... , 3n 中選擇了 n + 1 個不同的整數，則這些整數中總是存在兩個相差不超過 2。

Solution：

Ch 3, problem 5.

11. (10 points) Use combinatorial reasoning (組合解釋) to prove the identity.

$$1\binom{n}{1} + 2\binom{n}{2} + ... + n\binom{n}{n} = n2^{n-1}$$

p.s. 代數證明全對給一半分數，沒有部份分。

Solution：

Ch 5, eq. (5.8) 上課有證！

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	10	10	10	10	10	10	10	10	10	10	110
Score:												