應數二離散數學 2024 春, 期末考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 12 頁 (包含封面)，有 12 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Find the (ordinary) generating function for the infinite sequence $h_0, h_1, h_2, ...$ defined by $h_n = n + 5^n$.

   Answer: $\dfrac{x}{(1-x)^2} + \dfrac{1}{1-5x}$

   Solution：

   From Ch 7.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ...$$

$$\begin{aligned}
\frac{x}{(1-x)^2} &= x\frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= x\frac{d}{dx}\left(1 + x + x^2 + x^3 + ... + x^n + ...\right) \\
&= x + 2x^2 + 3x^3 + ... + nx^n + ... \\
&= \sum_{n=0}^{\infty} nx^n
\end{aligned}$$

$$\begin{aligned}
\frac{1}{1-5x} &= 1 + 5x + (5x)^2 + (5x)^3 + ... + (5x)^n + ... \\
&= \sum_{n=0}^{\infty} 5^n x^n
\end{aligned}$$

$$\frac{x}{(1-x)^2} + \frac{1}{1-5x} = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} 5^n x^n = \sum_{n=0}^{\infty} (n + 5^n)x^n$$

2. (10 points) Determine the number of ways to place six non-attacking rooks on the following 6-by-6 board, with forbidden positions as shown.

x	x				
	x	x			
x		x			
				x	
				x	x

Answer: $6! - 9 \times 5! + 28 \times 4! - 35 \times 3! + 15 \times 2! - 2 \times 1! = 130$.

Solution：

Using the Principle of Inclusion-Exclusion (6-4).

$$F_1 : \begin{array}{|c|c|c|}\hline \text{x} & \text{x} & \\ \hline & \text{x} & \text{x} \\ \hline \text{x} & & \text{x} \\ \hline \end{array}, \quad F_2 : \begin{array}{|c|c|}\hline \text{x} & \\ \hline \text{x} & \text{x} \\ \hline \end{array}$$

$$(1 + 6x + 9x^2 + 2x^3)(1 + 3x + x^2) = (1 + 9x + 28x^2 + 35x^3 + 15x^4 + 2x^5)$$

3. (10 points) Let p_n^s equal the number of self-conjugate partitions of n. Find p_{17}^s. *Hint:* By Theorem 8.3.2, let p_n^t be the number of partitions of n into distinct odd parts. Then $p_n^s = p_n^t$.

Answer: $p_{17}^s =$ 5

Solution：

p_{17}^t	p_{17}^s
17	9+1+1+1+1+1+1+1+1
13+3+1	7+3+3+1+1+1+1
11+5+1	6+4+3+2+1+1
9+7+1	5+5+3+2+2
9+5+3	5+4+4+3+1

,

$$\begin{array}{llllll} 5 & o & o & o & o & o \\ 5 & o & o & o & o & o \\ 3 & o & o & o & & \\ 2 & o & o & & & \\ 2 & o & o & & & \end{array}, \qquad \begin{array}{llllll} 5 & o & o & o & o & o \\ 4 & o & o & o & o & \\ 4 & o & o & o & o & \\ 3 & o & o & o & & \\ 1 & o & & & & \end{array},$$

$$\begin{array}{llllllllll} 9 & o & o & o & o & o & o & o & o & o \\ 1 & o \\ 1 & o \\ 1 & o \\ 1 & o \\ 1 & o \\ 1 & o \\ 1 & o \\ 1 & o \end{array}, \qquad \begin{array}{llllllll} 7 & o & o & o & o & o & o & o \\ 3 & o & o & o \\ 3 & o & o & o \\ 1 & o \\ 1 & o \\ 1 & o \\ 1 & o \end{array}, \qquad \begin{array}{lllllll} 6 & o & o & o & o & o & o \\ 4 & o & o & o & o \\ 3 & o & o & o \\ 2 & o & o \\ 1 & o \\ 1 & o \end{array},$$

4. (10 points) Let n be a positive integer. Let P_n^o be the set of partitions of n into odd parts, and let P_n^d be the set of partitions of n into distinct parts. In textbook, we establish a one-to-one correspondence between the two types of partitions. Then $|P_n^o| = |P_n^d|$. Please find the following corresponding partitions.

(a) the partition λ_1 : $351 = 3^{13}7^{23}9^{4}23^{5} \in P_n^o$ will corresponding to $\lambda_2 \in P_n^d$.

$\lambda_2 =$ $\underline{3+12+24+7+14+28+112+36+23+92}$.

(b) the partition τ_1 : $310 = 2+8+15+25+20+40+200 \in P_n^d$ will corresponding to $\tau_2 \in P_n^o$.

$\tau_2 =$ $\underline{1^{10}5^{12}15^{1}25^{9}}$.

Solution：

(a) $3\times(1+4+8)+7\times(1+2+4+16)+9\times(4)+23\times(1+4) = 3+12+24+7+14+28+112+36+23+92$

(b) $2+8+15+25+20+40+200 = 1\times 2+1\times 8+15\times 1+25\times 1+5\times 4+5\times 8+25\times 8 = 1\times(2+8)+15+25\times(1+8)+5\times(4+8)$

5. (10 points) The general term h_n of a sequence is a polynomial in n. If the first few elements are $5, 7, 23, 71, 169, 335, 587, 943, 1421, ...$, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.(不需化簡)

Answer: $h_n = $ $5\binom{n}{0} + 2\binom{n}{1} + 14\binom{n}{2} + 18\binom{n}{3}$.

$\sum_{k=0}^{n} h_k = $ $5\binom{n+1}{1} + 2\binom{n+1}{2} + 14\binom{n+1}{3} + 18\binom{n+1}{4}$.

Solution：

5		7		23		71		169		335		...
	2		16		48		98		166		...	
		14		32		50		68		...		
			18		18		18		...			
				0		0		...				

6. (15 points) Solve the nonhomogeneous recurrence relation $h_n = 4h_{n-1} - 4h_{n-2} + 3n + 1$ with initial values $h_0 = 1$, $h_1 = 2$. 提示：你可以分成 homogeneous 跟 non-homogeneous 的兩部分算。

Answer: $h_n =$ $\underline{\quad -12 \times 2^n + 5n2^n + 3n + 13 \quad}$.

Solution：

Ch 7

non-homogeneous:

Let $h_n = an + b \;\Rightarrow\; (an + b) = 4(a(n-1) + b) - 4(a(n-2) + b) + 3n + 1 \;\Rightarrow\; a = 3, b = 13$

homogeneous:

$h_n - 4h_{n-1} + 4h_{n-2} = 0$

$x^2 - 4x + 4 = (x-3)^2 \;\Rightarrow\; x = 2$ (重根)

$h_n = c_2 2^n + c_3 n 2^n$.

exact solution:

$h_n = c_2 2^n + c_3 n 2^n + 3n + 13$ 代回初始值

$$1 = h_0 = c_2 2^0 + c_3 0 \times 2^0 + 3 \times 0 + 13 = c_2 + 13 \;\Rightarrow\; c_2 = -12$$
$$2 = h_1 = c_2 2^1 + c_3 1 \times 2^1 + 3 \times 1 + 13 = 2c_2 + 2c_3 + 3 + 13 \;\Rightarrow\; c_3 = 5$$

$$h_n = -12 \times 2^n + 5n2^n + 3n + 13$$

7. (10 points) Use generating functions to determine the number of integral solutions of the equation

$$x_1 + 5x_2 + x_3 + x_4 = n,$$

that satisfy

$$0 \le x_1 \le 4, 0 \le x_2,\ 6 \le x_3,\ \ -2 \le x_4$$

Answer: $h_n = \binom{n-2}{2}$ **if** $n \ge 4$**, and** $h_n = 0$ **if** $n < 4$.

Solution：

It is better to use the ordinary type of generating function to solve this problem.

$\boxed{\text{Method 1}}$

The generating function is

$$\begin{aligned}\sum_{n=0}^{\infty} h_n x^n &= (1+x+x^2+x^3+x^4)(1+x^5+x^{10}+...)(x^6+x^7+x^8+...)(x^{-2}+x^{-1}+1+x+x^2+...)\\ &= \frac{1-x^5}{1-x} \times \frac{1}{1-x^5} \times \frac{x^6}{1-x} \times \frac{x^{-2}}{1-x} = \frac{x^4}{(1-x)^3}\\ &= x^4 \sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n = \sum_{n=0}^{\infty} \binom{n+2}{2} x^{n+4} = \sum_{n=4}^{\infty} \binom{n-2}{2} x^n\end{aligned}$$

$\boxed{\text{Method 2}}$

原式可改為 $y_1 = x + 1,\ y_2 = x_2,\ y_3 = x_3 - 6,\ y_4 = x_4 + 2$

$$y_1 + 5y_2 + y_3 + y_4 = x_1 + 5x_2 + (x_3 - 6) + (x_4 + 2) = \textbf{n-4},$$

that satisfy

$$0 \le y_1 \le 4, 0 \le y_2,\ 0 \le y_3,\ 0 \le y_4$$

The generating function is

$$\begin{aligned}\sum_{n=0}^{\infty} h_n x^n &= (1+x+x^2+x^3+x^4)(1+x^5+x^{10}+...)(1+x+x^2+...)(1+x+x^2+...)\\ &= \frac{1-x^5}{1-x} \times \frac{1}{1-x^5} \times \frac{1}{1-x} \times \frac{1}{1-x} = \frac{1}{(1-x)^3}\\ &= \sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n\end{aligned}$$

注意：h_n **是** x^{n-4} **的係數，所以是** $\binom{(n-4)+2}{2} = \binom{n-2}{2}$**。而且** $h_n = 0$ if $n < 4$**。**

8. (10 points) Let h_n denote the number of n-digit numbers with all digits at least 3, such that 4 and 7 each occur an even number of times, and 5 and 9 each occur at least once, there being no restriction on the rest digits. Determine the generating function $g(x)$ for the sequence $h_0, h_1, h_2, ...$ and then find a simple formula for h_n.

令 h_n 表示確定所有位數至少為 3 的 n 位數的數量，其中 4 和 7 都出現偶數次，且 5 和 9 至少各出現一次，剩下的數字沒有任何限制。確定序列 $h_0, h_1, h_2, ...$ 的生成函數 $g(x)$ ，並以此找到 h_n 的簡單公式。

Answer: (a) $g(x) =$ $\underline{\frac{1}{4}(e^{7x} - 2e^{6x} + 3e^{5x} - 4e^{4x} + 3e^{3x} - 2e^{2x} + e^{x})}$,

(b) $h_n =$ $\underline{\frac{1}{4}(7^n - 2\times 6^n + 3\times 5^n - 4\times 4^n + 3\times 3^n - 2\times 2^n + 1)}$.

Solution：

1. It is better to use the exponential type of generating function to solve this problem.
2. [all digits at least 3] = digits can be 3, 4, 5, 6, 7, 8, 9.
3. [there being no restriction on the rest digits] = there being no restriction on 3, 6, 8.

The generating function is

$$
\begin{aligned}
g(x) &= (1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!})^2(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+....)^2(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+....)^3\\
&= (\frac{(e^x+e^{-x})}{2})^2(e^x-1)^2(e^x)^3\\
&= \frac{1}{4}(e^{2x}+2+e^{-2x})(e^{2x}-2e^x+1)e^{3x}\\
&= \frac{1}{4}(e^{7x}-2e^{6x}+3e^{5x}-4e^{4x}+3e^{3x}-2e^{2x}+e^{x})
\end{aligned}
$$

9. (10 points) Prove that $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ and $s(n, n-1) = \binom{n}{2}$, where $s(n, k)$ is the Stirling numbers of the first kind and $S(n, k)$ is the second kind.

Solution：

This is Ch 8 problem 12(d) and 19(b).

推薦組合證明，但是若用代數證明，可以使用遞迴歸式。

10. (10 points) Let p_n is the number of partition of integer n. Prove that the partition function satisfies

$$p_n > p_{n-1} \ (n \geq 2)$$

Solution：

This is Ch 8 problem 20.

11. (10 points) Let D_n is the n^{th} derangement numbers. Prove that D_n is an even number if and only if n is an odd number.

Solution：

This is Ch 6 problem 21.

12. (10 points) Let m and n be nonnegative integers with $n \geq m$. There are m + n people in line to get into a theater for which admission is 50 cents. Of the m + n people, n have a 50-cent piece and m have a $1 dollar bill. The box office opens with an empty cash register. Show that the number of ways the people can line up so that change is available when needed is

$$\frac{n-m+1}{n+1}\binom{m+n}{m}$$

讓 m 和 n 為非負整數，且滿足 $n \geq m$。有 m + n 人排隊進入一個門票為 50 美分的劇院。在這 m + n 人中，有 n 人支付一個 50 美分的硬幣，而 m 人支付一張 1 美元的鈔票。售票亭在現金櫃檯是空的情況下開放。證明這些人排隊的方式中，確保需要時能提供找零的方式數為上式。

Solution：

Ch 8 problem 5.

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	10	10	10	10	10	15	10	10	10	10	10	10	125
Score:													