§6-3 Derangements (錯位排列)

Q: **At a party, 10 gentlemen check their hats.**
In how many ways can their hats be returned so that no gentleman gets the hat with which he arrived?

在一個派對上，有10位先生寄存了他們的帽子。有多少種歸還他們帽子的方式，讓每位先生都拿到不是自己寄存的那頂帽子？

Def: a derangement of $\{1,2,3,\dots,n\}$ is a permutation $\pi = \pi_1\pi_2\pi_3\cdots\pi_n$ s.t. $\forall i$, $\pi_i \neq i$

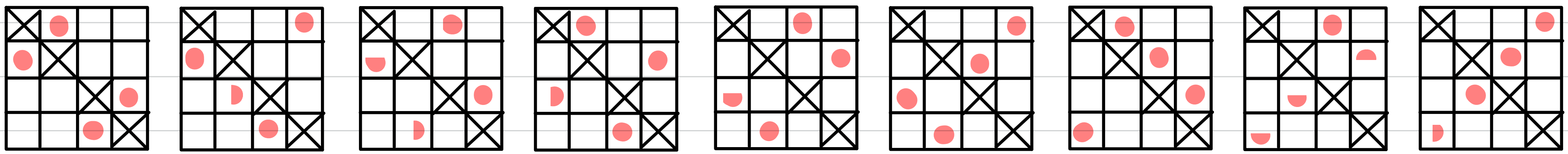
※ if $\pi_i = i$, we call it as a fixed point

$\therefore$ a derangement is a permutation without fixed point.

$n=2$ (1), ~~12~~, 21

$n=3$ (2), ~~123~~, ~~132~~, ~~213~~, 231, 312, ~~321~~

$n=4$ (9), 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321



Thm 6.3.1 : For $n \geq 1$, $D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}\right)$

pf.

using the inclusion-exclusion principle.

Let S = set of all permutation of $\{1, 2, 3, \ldots, n\} = S_n$

A_i = set of all permutation in S_n, with i is a fixed point.

$= \{ \pi = \pi_1 \pi_2 \pi_3 \cdots \pi_n \in S_n \mid \pi_i = i \}$

$$\therefore D_n = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n|$$
$$= |S| - \sum_i |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|$$
$$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n \binom{n}{n} 0!$$
$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

$\triangle$

A_1 : 1 _ _ ⋯ _ , A_2 : _ 2 _ _ ⋯ _ $\therefore (n-1)!$

$A_1 \cap A_2$: 1 2 _ _ ⋯ _ , $A_1 \cap A_3$: 1 _ 3 _ ⋯ _ $\therefore (n-2)!$

Recall : $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ $\therefore e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$

$\therefore D_n \approx n!\, e^{-1}$ or $\lim_{n \to \infty} \frac{n!}{D_n} = e$

$$n!\, e^{-1} = n\left((n-1)!\, e^{-1}\right) \approx n D_{n-1}$$

Thm (eq. (6.8)) : $D_n = n D_{n-1} + (-1)^n$

p.f.

$$n D_{n-1} + (-1)^n = n \cdot (n-1)! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^{n-1} \frac{1}{(n-1)!}\right) + (-1)^n \frac{n!}{n!}$$

$$= n! \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}\right) = D_n$$

Thm (eq. (6.6)) : $D_n = (n-1)(D_{n-1} + D_{n-2})$

p.f. ①

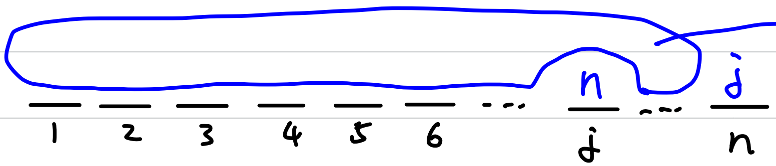
$$D_n = n D_{n-1} + (-1)^n = (n-1) D_{n-1} + (-1)^n + D_{n-1} = (n-1) D_{n-1} + (-1)^n + (n-1) D_{n-2} + (-1)^{n-1}$$

$$= (n-1)(D_{n-1} + D_{n-2})$$

p.f. ②

$\pi = \pi_1 \pi_2 \pi_3 \cdots \pi_n$, what is 「$\pi_n$」? △ $\pi_n \neq n$

i) $\pi_n = j$ & $\pi_j = n$ with $j = 1 \sim (n-1)$



$1 \sim (j-1), (j+1) \sim (n-1)$ total $(n-2)$ number ∴ D_{n-2}

$(n-2)$ position

there's $(n-1)$ difference 「j」 ∴ $(n-1) D_{n-2}$

(ii) $\pi_n = j$ & $\pi_j = l \neq n$ with $j = 1 \sim (n-1)$

$$\begin{array}{ccccccccc} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \cdots & \pi_{j-1} & l & \pi_{j+1} & \cdots & j \\ \hline 1 & 2 & 3 & 4 & & j-1 & j & j+1 & & n \end{array} \xrightarrow{\psi} \begin{array}{ccccccccc} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \cdots & \pi_{j-1} & \boxed{\overset{\times}{\frac{j}{j}}} & \pi_{j+1} & \cdots & l \\ \hline 1 & 2 & 3 & 4 & & j-1 & & j+1 & & n \end{array}$$

$1 \sim (j-1), (j+1) \sim n$ total $(n-1)$ number $\therefore D_{n-1}$

$(n-1)$ position

there's $(n-1)$ difference 「j」 $\therefore (n-1) D_{n-1}$

$^{\circ}$show $D_n = n D_{n-1} + (-1)^n$ by $D_n = (n-1)(D_{n-1} + D_{n-2})$

p.f.

$$\overset{-nD_{n-1}}{D_n} = (n-1)(D_{n-1} + D_{n-2}) - nD_{n-1} \Rightarrow D_n - nD_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2})$$

$$D_n - nD_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2}) = (-1)^2(D_{n-2} - (n-2)D_{n-3}) = (-1)^3(D_{n-3} - (n-3)D_{n-4})$$
$$= \cdots = (-1)^{n-2}(D_2 - 2D_1) = (-1)^{n-2} \times (1-0) = (-1)^{n-2} = (-1)^n$$

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

$D_2 = \{21\}$

$D_3 = \{312,\ 231\}$

$D_4 = \{2143,\ 2341,\ 2413,\ 3142,\ 3412,\ 3421,\ 4123,\ 4312,\ 4321\}$

i) $\pi_n = j.\ \pi_j = n$

$2143 \rightarrow \frac{2}{1}\ \frac{1}{2}$

$3412 \rightarrow \frac{3}{1}\ \frac{1}{3}$

$4321 \rightarrow \frac{3}{2}\ \frac{2}{3}$

$3 \times D_2$

(ii) $\pi_n = j,\ \pi_j = l \neq n$

$2341 \xrightarrow{4} 1342 \rightarrow \frac{3}{2}\ \frac{4}{3}\ \frac{2}{4} \rightsquigarrow 231$

$2413 \xrightarrow{4} 2431 \rightarrow \frac{2}{1}\ \frac{4}{2}\ \frac{1}{4} \rightsquigarrow 231$

$3142 \xrightarrow{4} 3241 \rightarrow \frac{3}{1}\ \frac{4}{3}\ \frac{1}{4} \rightsquigarrow 231$

$3421 \xrightarrow{4} 1423 \rightarrow \frac{4}{2}\ \frac{2}{3}\ \frac{3}{4} \rightsquigarrow 312$

$4123 \xrightarrow{4} 4132 \rightarrow \frac{4}{1}\ \frac{1}{2}\ \frac{2}{4} \rightsquigarrow 312$

$4312 \xrightarrow{4} 4213 \rightarrow \frac{4}{1}\ \frac{1}{3}\ \frac{3}{4} \rightsquigarrow 312$

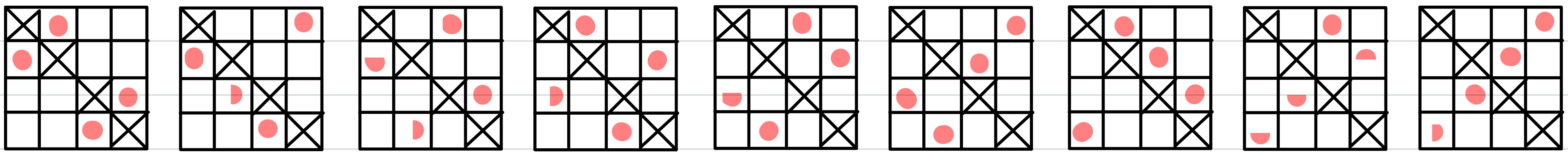
$3 \times D_1$

§ 6-4 Permutation with Forbidden (禁止) Position

Def: rook: (西洋棋：城堡，象棋：車) attack horizontally (橫) and vertically (直)

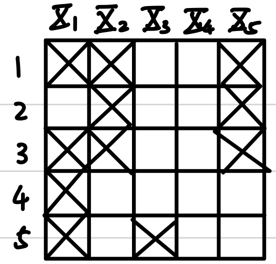
△ place n nonattacking rooks on a $n \times n$ chessboard with forbidden position
or permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ with sets $X_1, X_2, \dots, X_n$ s.t. $\forall i$, $\pi_i \notin X_i$

D_4: 2 1 4 3, 2 3 4 1, 2 4 1 3, 3 1 4 2, 3 4 1 2, 3 4 2 1, 4 1 2 3, 4 3 1 2, 4 3 2 1



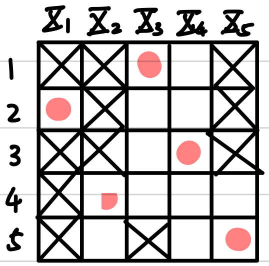
place n nonattacking rooks on a $n \times n$ chessboard, avoiding the diagonal
or permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ with sets $X_1, X_2, \dots, X_n$ s.t. $\forall i$, $\pi_i \notin X_i = \{i\}$

ex:

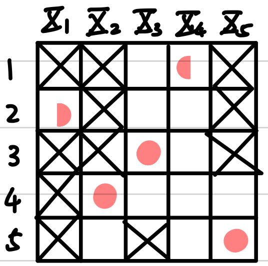


$S = \{1.2.3.4.5\}$

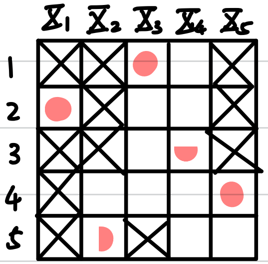
$X_1 = \{1.3.4.5\}$, $X_2 = \{1.2.3\}$, $X_3 = \{5\}$, $X_4 = \phi$, $X_5 = \{1.2.3\}$



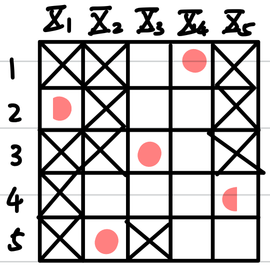
2 4 1 3 5



2 4 3 1 5



2 5 1 3 4



2 5 3 1 4

Question

$S = \{1.2.3.4.\dots, n\}$ $\quad X_1, X_2, X_3, X_4, \dots, X_n$: forbidden position.

sol:

$A_i = \{\pi = \pi_1\pi_2\pi_3 \cdots \pi_n \in S_n \mid \pi_i \in X_i\}$ or place rook in X_i in column i

$\therefore |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n|$

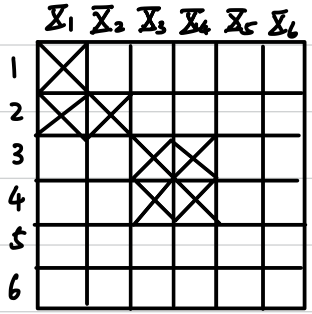
$= |S| - \sum_i |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|$

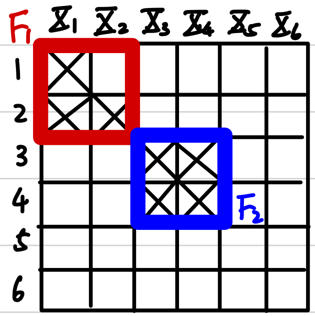
Δ $|A_i| = |X_i|(n-1)!$, $|A_i \cap A_j| = r_2 (n-2)!$, ..., $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k (n-k)!$

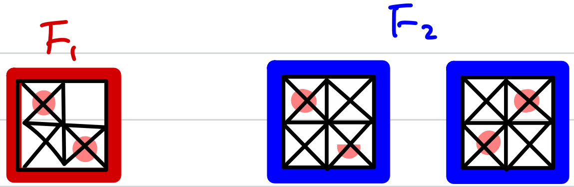
r_k: place k-nonattacking rooks in the 「Assign position」

$= n! - r_1(n-1)! + r_2(n-2)! - r_3(n-3)! + \cdots + (-1)^n r_n \cdot 0!$

ex:







$r_1 = 3 + 4 = 7$

F_1 , F_2

$r_2 = 3\times4 + 1 + 2 = 15$

$(F_1,F_2) , (F_1,F_1), (F_2,F_2)$

$r_3 = 1\times 4 + 3\times 2 = 10$

$(F_1,F_1,F_2) , (F_1,F_2,F_2)$

$r_4 = 1\times 2 = 2$

(F_1, F_1, F_2, F_2)

$r_5 = r_6 = 0$

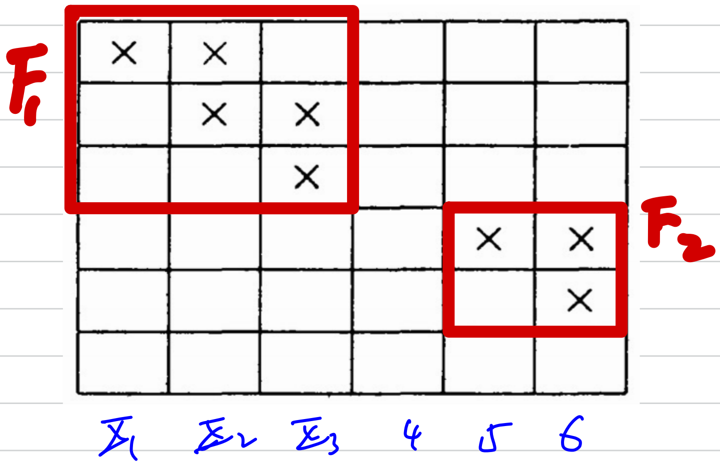
$\therefore |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n|$

$= n! - r_1(n-1)! + r_2(n-2)! - r_3(n-3)! + \cdots + (-1)^n r_n \cdot 0!$

$= 6! - 7\times5! + 15\times4! - 10\times3! + 2\times2! + 0$

$= 184$

①



$X_1 = \{1\}$

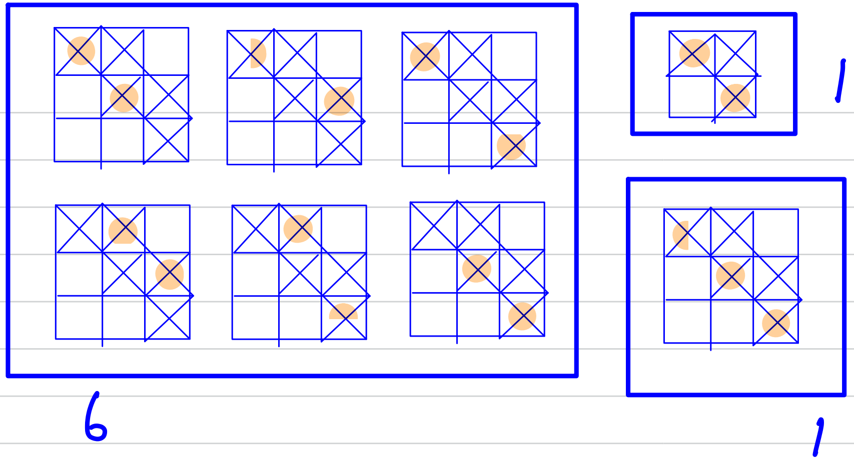
$X_2 = \{1, 2\}$

$X_3 = \{2, 3\}$

$X_4 = \phi$

$X_5 = \{4\}$

$X_6 = \{4, 5\}$



$$r_1 = \underset{(F_1)}{5} + \underset{(F_2)}{3} = 8$$

$$r_2 = \underset{(F_1, F_2)}{5 \times 3} + \underset{(F_1, F_1)}{6} + \underset{(F_2, F_2)}{1} = 22$$

$$r_3 = \underset{(F_1 \times 3)}{1} + \underset{(F_1 \times 2,\, F_2)}{6 \times 3} + \underset{(F_1,\, F_2 \times 2)}{5 \times 1} = 24$$

$$r_4 = \underset{(F_1 \times 3,\, F_2)}{1 \times 3} + \underset{(F_1 \times 2,\, F_2 \times 2)}{6 \times 1} = 9$$

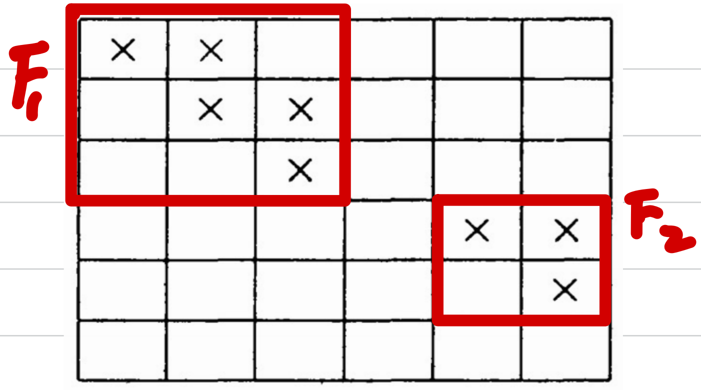
$$r_5 = \underset{(F_1 \times 3,\, F_2 \times 2)}{1 \times 1} = 1$$

$$\therefore |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n|$$

$$= n! - r_1 (n-1)! + r_2 (n-2)! - r_3 (n-3)! + \cdots + (-1)^n r_n \cdot 0!$$

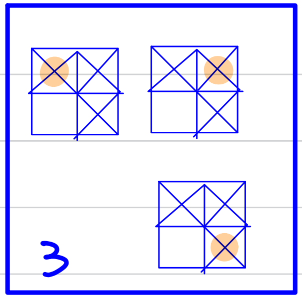
$$= 6! - 8 \times 5! + 22 \times 4! - 24 \times 3! + 9 \times 2! + 1 \times 1! - 0$$

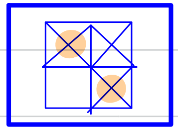
2



X_1 X_2 X_3 4 5 6

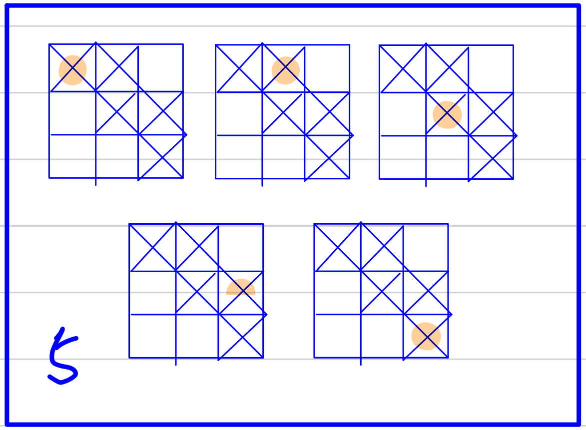
$X_1 = [1]$
$X_2 = [1, 2]$
$X_3 = [2, 3]$
$X_4 = \phi$
$X_5 = [4]$
$X_6 = [4, 5]$

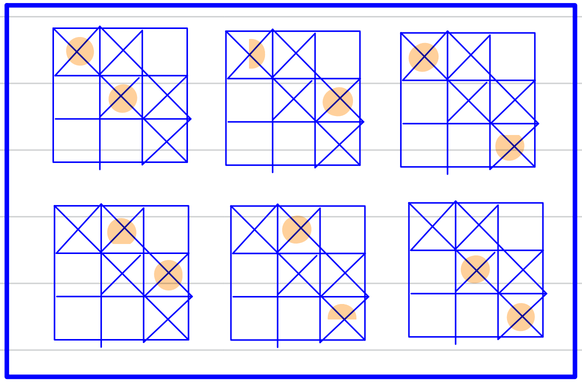




1

$\Rightarrow 1 + 3X + X^2$





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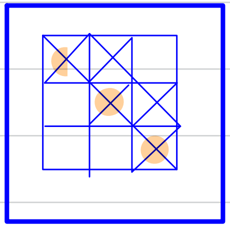

1

$\Rightarrow 1 + 5X + 6X^2 + X^3$

$$(1 + 3X + X^2)(1 + 5X + 6X^2 + X^3) = \underbrace{1}_{r_0} + \underbrace{8}_{r_1}X + \underbrace{22}_{r_2}X^2 + \underbrace{24}_{r_3}X^3 + \underbrace{9}_{r_4}X^4 + \underbrace{}_{r_5}X^5 + \underbrace{0}_{r_6}X^6$$

$\therefore |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n|$

$= n! - r_1(n-1)! + r_2(n-2)! - r_3(n-3)! + \cdots + (-1)^n r_n \cdot 0!$

$= 6! - 8 \times 5! + 22 \times 4! - 24 \times 3! + 9 \times 2! + 1 \times 1! - 0$