CATALAN NUMBER

Lemma 1.

$$\binom{1/2}{n} = \left(\frac{1}{2}\right)^{2n-1} (-1)^{n-1} \frac{1}{n} \binom{2n-2}{n-1}$$

Proof.

$$\begin{aligned}\binom{1/2}{n} &= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-3}{2})}{n!} = \left(\frac{1}{2}\right)^{n} (-1)^{n-1} \frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{n!} \\ &= \left(\frac{1}{2}\right)^{n} (-1)^{n-1} \frac{(2n-2)!}{n! 2 \times 4 \times \cdots \times (2n-2)} = \left(\frac{1}{2}\right)^{2n-1} (-1)^{n-1} \frac{1}{n} \frac{(2n-2)!}{(n-1)!(n-1)!} \\ &= \left(\frac{1}{2}\right)^{2n-1} (-1)^{n-1} \frac{1}{n} \binom{2n-2}{n-1}\end{aligned}$$

□

Theorem 2. *c_n is the n^{th} Catalan number, where $c_n = \frac{1}{n+1}\binom{2n}{n}$.*

Proof. Let $C(t) = \sum_{n=0}^{\infty} c_n t^n$ and we have

$$C(t) = t^0 + t \times C(t) \times C(t) = 1 + tC^2(t)$$

Solve the above equation, we have

$$C(t) = \frac{1 \pm \sqrt{1-4t}}{2t}$$

Note that

$$\begin{aligned}\sqrt{1-4t} = (1-4t)^{\frac{1}{2}} &= \sum_{n \geq 0} \binom{1/2}{n} (-4t)^n = 1 + \sum_{n \geq 1} \binom{1/2}{n} (-4t)^n \\ &= 1 + \sum_{n \geq 1} \left(\frac{1}{2}\right)^{2n-1} (-1)^{n-1} \frac{1}{n} \binom{2n-2}{n-1} (-4t)^n \\ &= 1 + \sum_{n \geq 1} (-2) \frac{1}{n} \binom{2n-2}{n-1} t^n.\end{aligned}$$

The plus sign makes no sense since

$$\frac{1+\sqrt{1-4t}}{2t} = \frac{1 + \left(1 + \sum_{n \geq 1} \frac{-2}{n} \binom{2n-2}{n-1} t^n\right)}{2t} = \frac{1}{t} - 1 - t + \cdots.$$

The minus sign gives

$$\begin{aligned}\frac{1-\sqrt{1-4t}}{2t} &= \frac{1 - \left(1 + \sum_{n \geq 1} \frac{-2}{n} \binom{2n-2}{n-1} t^n\right)}{2t} = \frac{\sum_{n \geq 1} \frac{2}{n} \binom{2n-2}{n-1} t^n}{2t} \\ &= \sum_{n \geq 1} \frac{1}{n} \binom{2n-2}{n-1} t^{n-1} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} t^n\end{aligned}$$

which results that $c_n = \frac{1}{n+1}\binom{2n}{n}$. □