Counting and Recounting: The Aftermath

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In my recent article (The Mathematical Intelligencer, 5.4 (1983), p. 21), I ended by challenging readers to provide a combinatorial proof to the identity:

$$4^n = \sum_{r=0}^{n} \binom{2r}{r} \binom{2n-2r}{n-r}.$$

This is a recount of the letters that I received from readers who continued where I left off by offering solutions to the problem.

The problem is not that new. P. Erdös, on reading the article, was quick to point out to me that Hungarian mathematicians tackled it in the thirties: P. Veress proposing and G. Hajos solving it. In his letter to me I. Gessel (M.I.T.) has given a survey of the more recent history of the problem. Proofs were published by D. Kleitman (Studies in Applied Mathematics 54. (1975), also by his student D. J. Kwiatowski (Ph.D. Thesis, MIT, 1975). It also found its way into texts (Feller, Mohanty).

In addition, I received solutions by A. Bondesen (Royal Danish School of Educational Studies, Copenhagen), K. Grünbaum (Roskilde Universitetscenter, Denmark), J. Hofbauer, jointly with N. Fulwick (Universität, Wien, Austria), D. Zeilberger (Drexel University, Philadelphia), and verbally from C. Pearce (Adelaide University), directly after reading the article.

All solutions are based, with some variations, on the count of lattice paths, or equivalently (1,0) sequences. Figure 1 is used to illustrate the simplest version. It represents a two-dimensional coordinate lattice, or a network of streets running East and North. We consider paths of length $2n$, beginning at O, proceeding in unit steps, heading East or North. It is clear that there are 2^{2n} ways in which a lattice point on the boundary AB can be reached. This gives the left-hand side of the identity.

Counting in a different way, assume that the last crossing of a path with the NE line (OM on the diagram) is at K(k,k), which of course may coincide with O or M. It is easy to see that there are $\binom{2k}{k}$ possible paths from O to K. Assuming for the moment that the number of ways the remaining $2n - 2k$ steps, (avoiding OM) may be taken, is similarly $\binom{2n-2k}{n-k}$, we obtain the desired right-hand side:

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k}.$$

The last statement, used to establish the identity, implies that the number of paths of length 2k, finishing at K (Fig. 2(a)), is the same as the number of paths of the same length, not touching the line OK (Fig. 2(b)). This is not obvious, but can be verified by manipulations involving binomial sums. An alternative geometric argument, coming from I. Gessel, is briefly sketched here.

As an intermediate step, consider paths of type shown in Fig. 2(c): these are completely in the upper half of the region, but may touch the line OK.

A path of type (a) may be transformed via (c) into type (b). The dotted line, drawn in Fig. 2(a), is a "tangent" to the path, parallel to OK and touching it for the first time at the extremity E. The transformation from (a) to (c) is done in three steps: cutting the path at E, translating the segment EK parallel to itself, bringing E to O and K to a point K′, and finally fitting the OE segment, by placing the end-point E at K′ and turning the segment to exchange vertical and horizontal directions, the image of the end originally at O coming to be the end of the path thus spliced together. A path of type (c) is thus obtained.

To get from (c) to (b) is necessary only if the path touches OK. In that case the horizontal unit-segment preceding the contact is turned vertical, the segment following it is shifted parallel to itself to the loose end, and finally this new path is reflected in OK into the lower region.

It can be shown easily that these "cutter, fitter, turner" operations from (a) to (c) and (c) to (b) have unique inverses (the reflection about OK in the second transformation ensures this). The composition of the two transformations gives a bijective map from (a) to (b).

A. Bondesen sent in an appealing variation of the theme of path-counting.

Fig. (3) represents "Polya-town" (Pascal triangle in disguise), the thick lines its streets, the circles, indexed by binomial coefficients, its corners. The thin lines with circles are the streets and corners of an under-

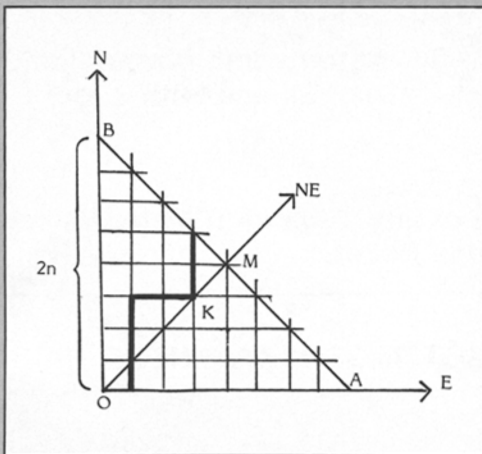

Figure 1

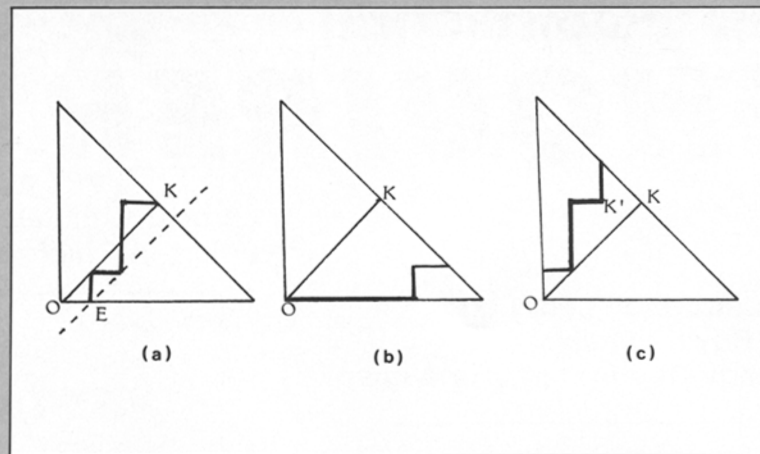

Figure 2

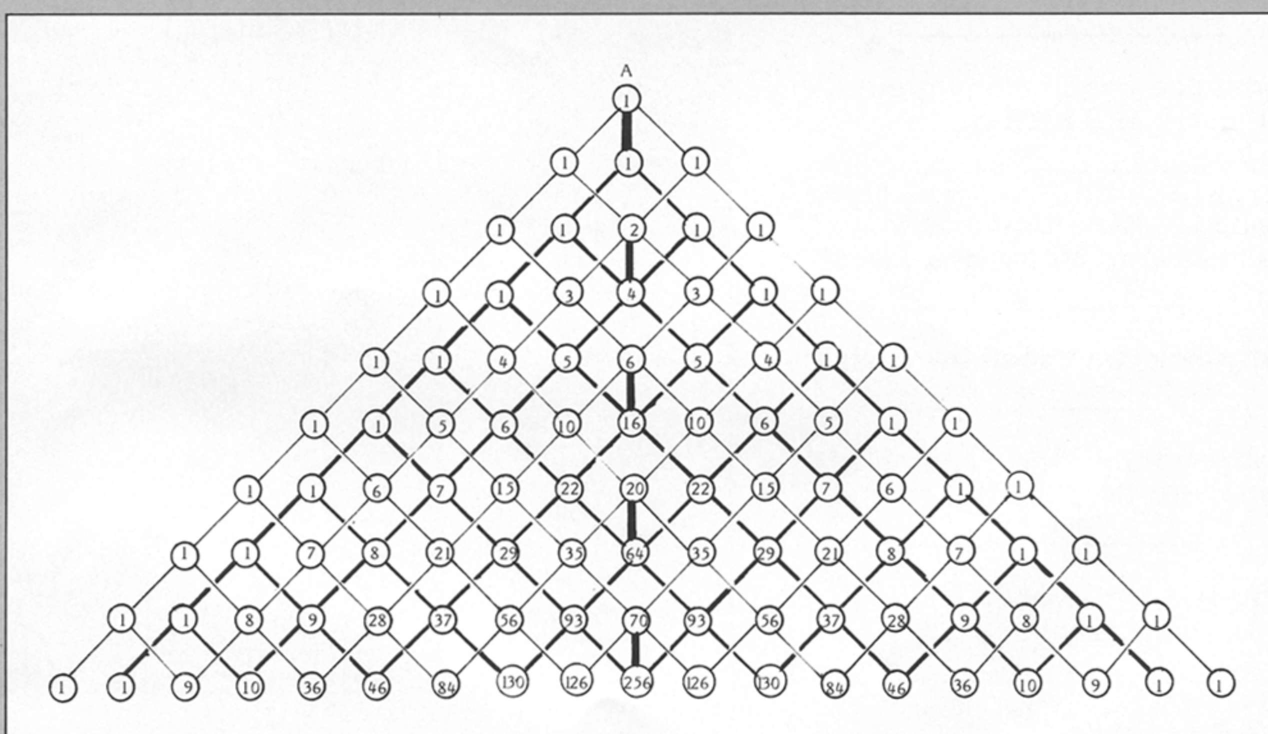

Figure 3

ground satellite town, while the heavy lines in the middle are ramps joining the two cities. The numbers indexing the corners of the satellite town are marked by the symbol $\widehat{\binom{n}{k}}$, defined as sums of binomial coefficients belonging to points of Polya-town in the same row, as the satellite-town point considered and between it and the nearer river drive. It is shown that

$$\widehat{\binom{n-1}{k-1}} + \widehat{\binom{n-1}{k}} = \widehat{\binom{n}{k}} + \delta^n_{2k}\binom{n}{k}$$

(δ the Kronecker symbol). The desired identity is interpreted as two counts of the shortest paths from A to the satellite point: $\widehat{\binom{2n}{n}} = 2^{2n}$, using streets of either town together with the ramps.

We are now a (lattice)-step nearer to conjecturing that combinatorial identities can be proved combinatorially.

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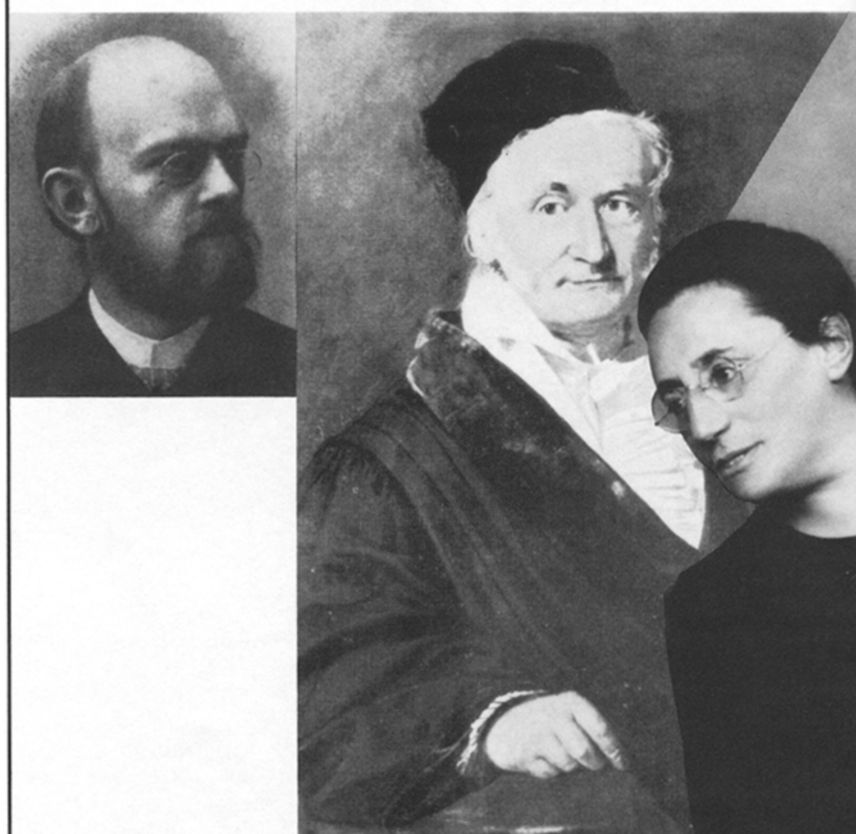

Hilbert's 24th Problem????????????

Big $$$$ in old colloquium notices——Was Ramanujan an E.T.?——Strange attractors stole my wife!!

Fermat speaks from grave: "I think I missed a factor of two; hope there's been no inconvenience!"

My life with Mobius TV only tells one side!!

Aus Heim und Welt