數學二離散數學 2025 秋, 期末考 **解答**

學號: ______________________________, 姓名: ______________________________

本次考試共有 9 頁 (包含封面)，有 11 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Consider the following 6-by-6 board with forbidden positions (marked with x). Determine the number of ways to place six non-attacking rooks on the allowed positions. (不需化簡)

x	x				
x		x			
		x			
				x	
				x	x

Answer: $1 \times 6! - 8 \times 5! + 22 \times 4! - 24 \times 3! + 9 \times 2! - 1 \times 1! = 161$

Solution：

Using the Principle of Inclusion-Exclusion (6-4).

$$F_1 : \begin{array}{|c|c|c|} \hline \text{x} & \text{x} & \\ \hline \text{x} & & \text{x} \\ \hline & & \text{x} \\ \hline \end{array}, \quad F_2 : \begin{array}{|c|c|} \hline \text{x} & \\ \hline \text{x} & \text{x} \\ \hline \end{array}$$

$$R(x, F_1) = 1 + 5x + 6x^2 + x^3, \ R(x, F_2) = 1 + 3x + x^2$$

Since F_1 and F_2 are disjoint, the rook polynomial for the entire forbidden board B is:

$$(1 + 5x + 6x^2 + x^3)(1 + 3x + x^2) = (1 + 8x + 22x^2 + 24x^3 + 9x^4 + 1x^5)$$

$$N = \sum_{k=0}^{6} (-1)^k r_k (6-k)!$$

2. (10 points) The general term h_n of a sequence is a polynomial in n. If the first few elements are $-1, 4, 21, 104, 355, 924, 2009, 3856, 6759, 11060, 17149.....$, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.(不需化簡)

Answer: $h_n =$ $-1\binom{n}{0} + 5\binom{n}{1} + 12\binom{n}{2} + 54\binom{n}{3} + 48\binom{n}{4} = 2n^4 - 3n^3 + n^2 + 5n - 1$.

$\sum_{k=0}^{n} h_k =$ $-1\binom{n+1}{1} + 5\binom{n+1}{2} + 12\binom{n+1}{3} + 54\binom{n+1}{4} + 48\binom{n+1}{5}$.

Solution：

n	0	1	2	3	4	5
h_n	**-1**	4	21	104	355	924
Δh_n	**5**	17	83	251	569	
$\Delta^2 h_n$	**12**	66	168	318		
$\Delta^3 h_n$	**54**	102	150			
$\Delta^4 h_n$	**48**	48				
$\Delta^5 h_n$	**0**					

3. (10 points) Solve the nonhomogeneous recurrence relation $h_n = 6h_{n-1} - 9h_{n-2} + 2n + 1$ with initial values $h_0 = 2$, $h_1 = 7$.

   (a) The characteristic roots are: $r =$ **3 (with multiplicity 2)** .

   (b) The form of the particular solution is $h_n^{(p)} =$ $an + b,\ a, b \in \mathbb{R}$, **or** $\frac{1}{2}n + \frac{7}{4}$.

   (c) The general solution is $h_n =$ $\left(\frac{1}{4} + \frac{4}{3}n\right)3^n + \frac{1}{2}n + \frac{7}{4}$.

   Solution：

   Ch 7

   The characteristic equation is:

$$r^2 - 6r + 9 = 0 \implies (r-3)^2 = 0 \implies r = 3 \quad \text{(with multiplicity 2)}$$

   (b) Particular Solution

$$h_n^{(p)} = An + B$$

   Substituting this into the recurrence relation:

$$(An + B) = 6(A(n-1) + B) - 9(A(n-2) + B) + 2n + 1 = -3An + (12A - 3B) + 2n + 1$$

$$\implies A = \frac{1}{2},\ B = \frac{7}{4}, \quad \implies h_n^{(p)} = \frac{1}{2}n + \frac{7}{4}$$

   (c) General Solution is the sum of the homogeneous solution and the particular solution:

$$h_n = (C_1 + C_2 n)3^n + \frac{1}{2}n + \frac{7}{4}$$

   Using initial conditions:

   - $h_0 = 2 \implies C_1 + \frac{7}{4} = 2 \implies C_1 = \frac{1}{4}$
   - $h_1 = 7 \implies (C_1 + C_2)3 + \frac{1}{2} + \frac{7}{4} = 7 \implies 3(\frac{1}{4} + C_2) + \frac{9}{4} = 7 \implies 3C_2 + 3 = 7 \implies C_2 = \frac{4}{3}$

   The final general solution is:

$$h_n = \left(\frac{1}{4} + \frac{4}{3}n\right)3^n + \frac{1}{2}n + \frac{7}{4}$$

4. (10 points) Solve the recurrence relation $h_n = 3h_{n-1} - 2h_{n-2}$ with initial values $h_0 = 1$, $h_1 = 4$.

(a) The characteristic roots are: $r =$ **1, 2** .

(b) The solution is $h_n =$ $-2 + 3 \cdot 2^n$.

Solution：

The characteristic equation is:

$$0 = r^2 - 3r + 2 = (r-1)(r-2), \implies r = 1, 2$$

The general solution for distinct real roots is:

$$h_n = C_1(1)^n + C_2(2)^n = C_1 + C_2 \cdot 2^n$$

- $h_0 = 1 \implies C_1 + C_2 = 1$
- $h_1 = 4 \implies C_1 + 2C_2 = 4, \implies C_2 = 3, C_1 = -2$

The final solution is:

$$h_n = -2 + 3 \cdot 2^n$$

如果用生成函數的做法，(a) 小題可以跳過不做。

5. (10 points) Determine the number of integral solutions of the equation

$$x_1 + 6x_2 + x_3 + x_4 = n, \quad \text{that satisfy } 0 \le x_1 \le 5,\ 0 \le x_2,\ 6 \le x_3,\ -2 \le x_4$$

(a) The generating function $G(x) =$ $\frac{x^4}{(1-x)^3}$.

(b) The coefficient of x^n in the function $G(x) =$ $h_n = \binom{n-2}{2} = \frac{(n-2)(n-3)}{2}$.

Solution：

$$G(x) = \left(\frac{1-x^6}{1-x}\right)\left(\frac{1}{1-x^6}\right)\left(\frac{x^6}{1-x}\right)\left(\frac{x^{-2}}{1-x}\right) = \frac{x^4}{(1-x)^3}$$

$$\because \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty}\binom{n+2}{2}x^n, \quad \therefore [x^n]G(x) = \binom{(n-4)+2}{2} = \binom{n-2}{2}$$

注意一下，平常答案應該是 $h_n = \begin{cases} 0, & if\ n < 4 \\ \binom{n-2}{2}, & if\ n \ge 4 \end{cases}$. 但是反正 $\binom{n-2}{2}$ 在 $n < 4$ 時，本來就是 0，就不用加條件了。

6. (10 points) Let h_n denote the number of n-digit numbers with all digits at least 3, such that 7 occur an even number of times, 5, 6 each occur an odd number of times, and 9 occur at least once, there being no restriction on the rest digits. Determine the generating function $g(x)$ for the sequence $h_0, h_1, h_2, ...$ and then find a simple formula for h_n.

Answer: (a) $g(x) = \underline{\frac{1}{8}(e^{7x} - e^{6x} - e^{5x} + e^{4x} - e^{3x} + e^{2x} + e^x - 1)}$,

(b) $h_n = \underline{\begin{cases} \frac{1}{8}(7^n - 6^n - 5^n + 4^n - 3^n + 2^n + 1) & \textbf{if } n \geq 1 \\ \frac{1}{8}(7^0 - 6^0 - 5^0 + 4^0 - 3^0 + 2^0 + 1 - 1) = 0 & \textbf{if } n = 0 \end{cases}}$.

Solution：

1. It is better to use the exponential type of generating function to solve this problem.
2. [all digits at least 3] = digits can be 3, 4, 5, 6, 7, 8, 9.
3. [there being no restriction on the rest digits] = there being no restriction on 3, 4, 8.

(a) Generating Function $g(x)$ We use the exponential generating function (EGF) for each digit:

- Digit 7 (even times): $\frac{e^x+e^{-x}}{2}$
- Digits 5 and 6 (each odd times): $\left(\frac{e^x-e^{-x}}{2}\right)^2$
- Digit 9 (at least once): $(e^x - 1)$
- Digits 3, 4, 8 (no restriction): $(e^x)^3 = e^{3x}$

The total EGF is:

$$\begin{aligned} g_e(x) &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)^2 (e^x - 1)e^{3x} \\ &= \frac{1}{8}(e^{2x} - e^{-2x})(e^x - e^{-x})(e^x - 1)e^{3x} \\ &= \frac{1}{8}(e^{3x} - e^x - e^{-x} + e^{-3x})(e^{4x} - e^{3x}) \\ &= \frac{1}{8}(e^{7x} - e^{6x} - e^{5x} + e^{4x} - e^{3x} + e^{2x} + e^x - 1) \end{aligned}$$

(b) Formula for h_n The value h_n is the coefficient of $\frac{x^n}{n!}$ in $g_e(x)$. For $n \geq 1$:

$$h_n = \frac{1}{8}(7^n - 6^n - 5^n + 4^n - 3^n + 2^n + 1)$$

7. (10 points) Consider a I-by-n chessboard. Suppose we color each square of the chessboard with one of the two colors red and blue. Let h_n be the number of colorings in which no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then derive a formula for h_n .

Answer: recurrence relation: $h_n = h_{n-1} + h_{n-2}$**,** $h_0 = 1$ **and** $h_1 = 2$.

The formula for h_n: $h_n = f_{n+2}$.

Solution：

By construction $h_0 = 1$ and $h_1 = 2$. We now find h_n for $n \geq 2$. Consider a coloring of the $1 \times n$ chessboard. The first square is colored red or blue. If it is blue, then there are h_{n-1} ways to color the remaining $n-1$ squares. If it is red, then the second square is blue, and there are h_{n-2} ways to color the remaining n − 2 squares. Therefore $h_n = h_{n-1} + h_{n-2}$. Comparing the above data with the Fibonacci sequence we find $h_n = f_{n+2}$.

Formula for h_n: (寫成 Fibonacci 就行，沒有一定要算出來)

$$h_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right] = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^{n} + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^{n}$$

特別注意，寫遞迴式時，記得要寫上起始值。然後這題特別提醒了要寫遞迴的理由！

8. (10 points) (a) Write down a combinatorial model (組合模型) for the Catalan number. (請跟 (b) 小題不一樣的)

Solution：

Check ch 8-1，超多種的，可以是路徑、投票、正多邊形分三角數、樹... 等等。只需要給模型描述即可。

(b) Let $2n$ (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the n^{th} Catalan number C_n.

Solution：

Ch 8 problem 1.

這題最方便的做法，就是做一個 bijection，把這題的模型對應到 a 小題的。

9. (10 points) Let D_n is the n^{th} derangement numbers. Prove that D_n is an even number if and only if n is an odd number.

Solution：

This is Ch 6 problem 21.

Key: 利用以下式子配合數學歸納法，且初始值 $D_1 = 0,\ D_2 = 1$

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \ or\ D_n = nD_{n-1} + (-1)^n$$

10. (10 points) Prove that $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ and $s(n, n-1) = \binom{n}{2}$, where $s(n,k)$ is the Stirling numbers of the first kind and $S(n,k)$ is the second kind.

Solution：

This is Ch 8 problem 12(d) and 19(b).

推薦組合證明，但是若用代數證明，可以使用遞迴歸式。

Key: 使用定義做計算，不管是不是組合模型數都可以。

若使用組合模型數，則定義如下：

The Stirling number of the first kind $s(n,k)$ counts the number of ways to arrange n elements into k disjoint cycles.

The Stirling number of the second kind $S(n,k)$ counts the number of ways to partition a set of n elements into k non-empty subsets.

11. (20 points) Given a sequence: $h_0 = \binom{0}{2}$, $h_1 = \binom{1}{2}$, $h_2 = \binom{2}{2}, \ldots, h_n = \binom{n}{2}, \ldots$.

(a) Find the ordinary generating function of $\{h_n\}$ $\underline{\frac{x^2}{(1-x)^3}}$.

(b) Find the exponential generating function of $\{h_n\}$ $\underline{\frac{x^2e^x}{2}}$.

Solution：

Given $h_n = \binom{n}{2} = \frac{n(n-1)}{2}$ for $n \geq 2$, and $h_0 = h_1 = 0$.

(a) Ordinary Generating Function (OGF)

Using the generalized binomial theorem:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

For $k = 3$:

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n$$

Shifting the index by multiplying by x^2:

$$\frac{x^2}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n}{2} x^n$$

(b) Exponential Generating Function (EGF)

By definition:

$$E(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} \frac{x^n}{n!}$$

$$E(x) = \sum_{n=2}^{\infty} \frac{1}{2} \frac{x^n}{(n-2)!} = \frac{x^2}{2} \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} = \frac{x^2}{2} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \frac{x^2}{2} e^x$$

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	10	10	10	10	10	10	10	10	10	20	120
Score:												