應數二離散數學 2026 春, 第一次期中考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 9 頁 (包含封面)，有 13 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也可能得到零分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。　**敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。　**遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Find the sum $\sum_{k=0}^{n}\binom{n}{k}r^k$ for all n.

   Answer: $(r+1)^n$.

   Solution：

   By the Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.
   Let $x = r$ and $y = 1$. We obtain $\sum_{k=0}^{n}\binom{n}{k}r^k(1)^{n-k} = (r+1)^n$.

2. (10 points) How many ways are there to arrange the letters in the word "VISITING" such that no two I's are consecutive?

   Answer: $5! \times \binom{6}{3} = 120 \times 20 = 2400$.

   Solution：

   The word contains: V, S, T, N, G (5 distinct letters) and I, I, I (3 identical letters).
   1. Arrange the 5 distinct letters: $5! = 120$ ways.
   2. These 5 letters create 6 potential slots (including ends): _ V _ S _ T _ N _ G _
   3. Choose 3 slots out of 6 to place the I's: $\binom{6}{3} = 20$.
   Total: $120 \times 20 = 2400$.

   Solution：

   To apply PIE (Principle of Inclusion-Exclusion) rigorously, treat the three 'I's as distinct (I_1, I_2, I_3). Let S be the universal set of all permutations of the 8 distinct letters. Let P_1, P_2, P_3 be the properties that pairs (I_1, I_2), (I_2, I_3), and (I_1, I_3) are adjacent, respectively. Let A_i be the set of permutations satisfying property P_i.

   - $|S| = 8! = 40320$
   - $\sum |A_i| = \binom{3}{1} \times 2! \times 7! = 30240$
   - $\sum |A_i \cap A_j| = \binom{3}{2} \times 2! \times 6! = 4320$ (Blocks like $I_1I_2I_3$ or $I_3I_2I_1$)
   - $\sum |A_1 \cap A_2 \cap A_3| = 0$ (Impossible to form a cycle in a line)

   By Theorem 6.1.1, the number of valid arrangements with distinct 'I's is:

   $$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_1 \cap A_2 \cap A_3|$$

   $$= 40320 - 30240 + 4320 - 0 = 14400$$

   Since the three 'I's are identical, divide by 3! to remove the artificial ordering:

   $$\text{Total} = \frac{14400}{3!} = 2400$$

3. (10 points) Determine the number of integral solutions of the equation

$$x + y + z + w = 20$$

that satisfy

$$1 \leq x \leq 5, \quad -2 \leq y \leq 4, \; 3 \leq z \leq 8, \; 4 \leq w \leq 7.$$

Answer: $\binom{17}{3} - \left[\binom{12}{3} + \binom{10}{3} + \binom{11}{3} + \binom{13}{3}\right] + \left[\binom{5}{3} + \binom{6}{3} + \binom{8}{3} + \binom{4}{3} + \binom{6}{3} + \binom{7}{3}\right] = 34$.

Solution：

Perform change of variables: $x' = x - 1, y' = y + 2, z' = z - 3, w' = w - 4$.
The new equation is $x' + y' + z' + w' = 20 - 1 - (-2) - 3 - 4 = 14$.
Constraints: $0 \leq x' \leq 4, 0 \leq y' \leq 6, 0 \leq z' \leq 5, 0 \leq w' \leq 3$.
Total non-negative solutions $S = \binom{14+4-1}{4-1} = \binom{17}{3} = 680$.
Let A_1, A_2, A_3, A_4 be the sets of solutions violating the upper bounds of x', y', z', w' respectively:
$|A_1| = \binom{14-5+3}{3} = \binom{12}{3} = 220, |A_2| = \binom{10}{3} = 120, |A_3| = \binom{11}{3} = 165, |A_4| = \binom{13}{3} = 286.$
Intersections: $|A_1 \cap A_2| = \binom{14-5-7+3}{3} = \binom{5}{3} = 10, |A_1 \cap A_3| = \binom{7}{3} = 35 \ldots$
Applying Inclusion-Exclusion Principle:
$N = 680 - (220 + 120 + 165 + 286) + (10 + 20 + 56 + 4 + 20 + 35) = 34.$
Final Answer: **34** .

4. (10 points) Construct a permutation of $\{1, 2, \ldots, 9\}$ whose inversion sequences are 6, 2, 5, 4, 1, 2, 0, 1, 0.

The resulting permutation is: **7 5 2 9 6 4 1 3 8** .

Solution：

Using the insertion method (Algorithm II):
1. $a_9 = 0$: (9)
2. $a_8 = 1$: (9, 8)
3. $a_7 = 0$: (7, 9, 8)
4. $a_6 = 2$: (7, 9, 6, 8)
5. $a_5 = 1$: (7, 5, 9, 6, 8)
6. $a_4 = 4$: (7, 5, 9, 6, 4, 8)
7. $a_3 = 5$: (7, 5, 9, 6, 4, 3, 8)
8. $a_2 = 2$: (7, 5, 2, 9, 6, 4, 3, 8)
9. $a_1 = 6$: (7, 5, 2, 9, 6, 4, 1, 3, 8)

5. (10 points) A committee of 6 people is to be chosen from 9 men and 7 women. (a) How many ways if there are no restrictions? (b) How many ways if the committee must contain at least 4 women?

Answer: (a) $\binom{16}{6} = 8008$, (b) **1456** .

Solution：

(a) Total ways $= \binom{16}{6} = 8008$.
(b) We consider three cases for the number of women:
Case 1 (4 women, 2 men): $\binom{7}{4}\binom{9}{2} = 35 \times 36 = 1260$.
Case 2 (5 women, 1 man): $\binom{7}{5}\binom{9}{1} = 21 \times 9 = 189$.
Case 3 (6 women, 0 men): $\binom{7}{6}\binom{9}{0} = 7 \times 1 = 7$.
Total $= 1260 + 189 + 7 = 1456$.

6. (10 points) Find the position of the 3-subset $\{2,4,7\}$ of $\{1,2,3,4,5,6,7,8\}$ in lexicographic order.

Answer: $\binom{8}{3}-\binom{8-2}{3}-\binom{8-4}{2}-\binom{8-7}{1}=56-20-6-1=29$.

Solution：

The lexicographic position of a subset $\{a_1,a_2,a_3\}$ of $\{1,\ldots,n\}$ is given by:
Rank $=\binom{n}{3}-\binom{n-a_1}{3}-\binom{n-a_2}{2}-\binom{n-a_3}{1}$
Rank $=\binom{8}{3}-\binom{8-2}{3}-\binom{8-4}{2}-\binom{8-7}{1}$
Rank $=56-\binom{6}{3}-\binom{4}{2}-\binom{1}{1}=56-20-6-1=29.$
Final Answer: **29** .

7. (10 points) Find the coefficient of x^5 in the expansion of $(2-x+3x^2)^8$.

Answer: $\binom{8}{5,1,2}(2^5)(-1^1)(3^2)+\binom{8}{4,3,1}(2^4)(-1^3)(3^1)+\binom{8}{3,5,0}(2^3)(-1^5)(3^0)=-62272$.

Solution：

Possible (n_1,n_2,n_3) such that $n_1+n_2+n_3=8$ and $n_2+2n_3=5$:
1. $n_3=2\Rightarrow n_2=1,n_1=5$. Coeff: $\frac{8!}{5!1!2!}(2^5)(-1^1)(3^2)=-48384.$
2. $n_3=1\Rightarrow n_2=3,n_1=4$. Coeff: $\frac{8!}{4!3!1!}(2^4)(-1^3)(3^1)=-13440.$
3. $n_3=0\Rightarrow n_2=5,n_1=3$. Coeff: $\frac{8!}{3!5!0!}(2^3)(-1^5)(3^0)=-448.$

Total: -62272.

8. (10 points) Find the number of integers between 1 and 1000 (inclusive) that are not divisible by 4, 5, or 6.

Answer: 534 .

Solution：

Let S be the universal set of integers from 1 to 1000, so $|S| = 1000$. Let A_4, A_5, A_6 be the sets of integers in S that are divisible by 4, 5, and 6, respectively.

$$|A_4| = \lfloor \frac{1000}{4} \rfloor = 250,\ |A_5| = \lfloor \frac{1000}{5} \rfloor = 200,\ |A_6| = \lfloor \frac{1000}{6} \rfloor = 166$$

$$|A_4 \cap A_5| = |A_{20}| = \lfloor \frac{1000}{20} \rfloor = 50,\ |A_4 \cap A_6| = |A_{12}| = \lfloor \frac{1000}{12} \rfloor = 83,\ |A_5 \cap A_6| = |A_{30}| = \lfloor \frac{1000}{30} \rfloor = 33$$

$$|A_4 \cap A_5 \cap A_6| = |A_{60}| = \lfloor \frac{1000}{60} \rfloor = 16$$

$$\begin{aligned} |\overline{A_4} \cap \overline{A_5} \cap \overline{A_6}| &= |S| - \sum |A_i| + \sum |A_i \cap A_j| - |A_4 \cap A_5 \cap A_6| \\ &= 1000 - (250 + 200 + 166) + (50 + 83 + 33) - 16 = 534 \end{aligned}$$

9. (10 points) Find the number of derangements of $\{1, 2, 3, 4, 5\}$ that start with the number 5.

Answer: 11 .

Solution：

利用我上課給的式子 (6.6) 的組合證明，拆成兩種可能。

Method 1

Given that 1 is mapped to 5.

- Case 1: 5 is mapped back to 1. The remaining numbers $\{2, 3, 4\}$ must be deranged, which is $D_3 = 2$ ways.

  *Map elements $\{1, 2, 3\} \to 2, 3, 4$, then add 5 at pos 1 and 1 at pos 5.

- Case 2: 5 is NOT mapped to 1. This is equivalent to a derangement of 4 elements $\{2, 3, 4, 5\}$ where the forbidden position for 5 is 1. This is $D_4 = 9$ ways.

  * Set pos 1=5, move D_4's pos 1 to pos 5, others stay.

Total $= 2 + 9 = 11$.

Method 2

From D_3:

231 $\Rightarrow$ 53421
312 $\Rightarrow$ 54231

From D_4:

2143 $\Rightarrow$ 51432
2341 $\Rightarrow$ 53412
2413 $\Rightarrow$ 54132
3142 $\Rightarrow$ 51423
3412 $\Rightarrow$ 54123
3421 $\Rightarrow$ 54213
4123 $\Rightarrow$ 51234
4312 $\Rightarrow$ 53124
4321 $\Rightarrow$ 53214

10. (10 points) Given 150 integers $a_1, a_2, \ldots, a_{150}$, there exist integers r and s with $0 \le r < s \le 150$ such that $a_{r+1} + a_{r+2} + \cdots + a_s$ is divisible by 150. Prove this statement.

Solution：

Define partial sums $S_k = a_1 + a_2 + \cdots + a_k$ for $k = 1, \ldots, 150$, and let $S_0 = 0$.
Consider these 151 sums: $S_0, S_1, S_2, \ldots, S_{150}$.
According to the Pigeonhole Principle, when these 151 numbers are divided by 150, there are 150 possible remainders $\{0, 1, \ldots, 149\}$. Thus, at least two sums S_r and S_s $(0 \le r < s \le 150)$ must have the same remainder.
Therefore, $S_s - S_r = a_{r+1} + \cdots + a_s$ is divisible by 150.

11. (10 points) Show that among any $n+1$ integers chosen from $\{1, 2, \ldots, 2n\}$, there are always two numbers such that one divides the other.

Solution：

Write each chosen integer x_i as $x_i = 2^{k_i} \cdot m_i$, where m_i is odd. Since $x_i \leq 2n$, the odd part m_i must be in the set $\{1, 3, 5, \ldots, 2n-1\}$. This set has n elements. Since we chose $n+1$ numbers, by PHP, two numbers x_i, x_j must share the same odd part m. Thus $x_i = 2^{k_i} \cdot m$ and $x_j = 2^{k_j} \cdot m$. The one with the smaller power of 2 divides the other.

12. (10 points) Use combinatorial reasoning to prove the identity:

$$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}$$

Solution：

This is the Hockey-stick identity. (**我上課有證過！**)

RHS (Right-Hand Side): The total number of ways to simply choose n people from the $r+n+1$ people is directly given by $\binom{r+n+1}{n}$.

LHS (Left-Hand Side): Alternatively, we can count the selections by checking the people in numerical order to determine the **first person NOT chosen**:

- If Person 1 is NOT chosen: We must select all n people from the remaining $r+n$ people. Ways: $\binom{r+n}{n}$.
- If Person 1 is chosen, but Person 2 is NOT chosen: We need $n-1$ more people from the remaining $r+n-1$ people. Ways: $\binom{r+n-1}{n-1}$.
- If Persons 1 and 2 are chosen, but Person 3 is NOT chosen: We need $n-2$ more people from the remaining $r+n-2$ people. Ways: $\binom{r+n-2}{n-2}$.
- ...
- If Persons $1, 2, \ldots, n$ are chosen, but Person $n+1$ is NOT chosen: We need 0 more people from the remaining r people. Ways: $\binom{r}{0}$.

Equivalently, this approach partitions the combinations based on the **smallest numbered person who is NOT chosen**. Suppose the smallest unchosen person is number m (where $1 \leq m \leq n+1$). This means the first $m-1$ people are already selected for the team. We then need to choose the remaining $n-(m-1)$ people from the available $(r+n+1)-m$ people.

Letting $k = n-m+1$ (which means k ranges from n down to 0), the number of ways for each case perfectly matches $\binom{r+k}{k}$. Summing these disjoint cases over all possible values of k yields the LHS:

$$\sum_{k=0}^{n} \binom{r+k}{k}$$

13. (10 points) How many permutations of the multiset $\{1,1,2,2,3,3,4,4\}$ are there such that no two identical digits are adjacent? (沒有任何兩個相同數字相鄰)

Answer: 864 .

Solution：

Total permutations $|S| = \frac{8!}{2!2!2!2!} = 2520$.
Let P_i be the property that the two i's are adjacent.
$S_1 = \sum |P_i| = \binom{4}{1}\frac{7!}{2!2!2!} = 4 \times 630 = 2520$.
$S_2 = \sum |P_i \cap P_j| = \binom{4}{2}\frac{6!}{2!2!} = 6 \times 180 = 1080$.
$S_3 = \sum |P_i \cap P_j \cap P_k| = \binom{4}{3}\frac{5!}{2!} = 4 \times 60 = 240$.
$S_4 = |P_1 \cap P_2 \cap P_3 \cap P_4| = \binom{4}{4}4! = 24$.
By Inclusion-Exclusion Principle:
$N = 2520 - 2520 + 1080 - 240 + 24 = 864$.
Final Answer: **864** .

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
Points:	10	10	10	10	10	10	10	10	10	10	10	10	10	130
Score:														