應數二離散數學 2026 春, 期末考 **解答**

學號: ____________________________, 姓名: ____________________________

本次考試共有 13 頁 (包含封面)，有 13 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Consider the following 6-by-6 board with forbidden positions (marked with x). Determine the number of ways to place six non-attacking rooks on the allowed positions. (不需化簡)

x	x				
	x				
		x	x		
			x		
				x	x

Answer: $1 \times 6! - 8 \times 5! + 23 \times 4! - 28 \times 3! + 13 \times 2! - 2 \times 1! = 168$

Solution：

Using the Principle of Inclusion-Exclusion (6-4).

The forbidden board B can be decomposed into three disjoint sub-boards F_1, F_2, and F_3:

$$F_1 : \begin{array}{|c|c|}\hline \text{x} & \text{x} \\ \hline & \text{x} \\ \hline \end{array}, \quad F_2 : \begin{array}{|c|c|}\hline \text{x} & \text{x} \\ \hline & \text{x} \\ \hline \end{array}, \quad F_3 : \begin{array}{|c|c|}\hline \text{x} & \text{x} \\ \hline \end{array}$$

The rook polynomials for these sub-boards are:

$$R(x, F_1) = R(x, F_2) = 1 + 3x + x^2, \quad R(x, F_3) = 1 + 2x$$

Since F_1, F_2, and F_3 are disjoint, the rook polynomial for the entire forbidden board B is:

$$\begin{aligned} R(x, B) &= R(x, F_1) \times R(x, F_2) \times R(x, F_3) \\ &= (1 + 3x + x^2)^2(1 + 2x) \\ &= (1 + 6x + 11x^2 + 6x^3 + x^4)(1 + 2x) \\ &= 1 + 8x + 23x^2 + 28x^3 + 13x^4 + 2x^5 \end{aligned}$$

By the Principle of Inclusion-Exclusion, the number of ways to place 6 non-attacking rooks on the allowed positions is:

$$\begin{aligned} N &= \sum_{k=0}^{6} (-1)^k r_k (6-k)! \\ &= 1 \times 6! - 8 \times 5! + 23 \times 4! - 28 \times 3! + 13 \times 2! - 2 \times 1! \\ &= 168 \end{aligned}$$

2. (10 points) Given a sequence with $h_0 = 2,\ h_1 = 3,\ h_2 = 10$, and the third difference is constant: $\Delta^3 h_n = 18$. Determine h_3, the expression of h_n, and a formula for $\sum_{k=0}^{n} h_k$. (不需化簡)

Answer: $h_3 =$ $\underline{\ 41\ }$; $h_n =$ $\underline{2\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 18\binom{n}{3} = 3n^3 - 6n^2 + 4n + 2}$; $\sum_{k=0}^{n} h_k =$ $\underline{2\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 18\binom{n+1}{4}}$.

Solution：

Build the difference table; the leading ($n = 0$) diagonal gives the Newton coefficients.

n	0	1	2	3
h_n	**2**	3	10	41
Δh_n	**1**	7	31	
$\Delta^2 h_n$	**6**	24		
$\Delta^3 h_n$	**18**	18		

Since $\Delta^3 h_n = 18$ is constant, h_n is a cubic. From $\Delta^2 h_1 = \Delta^2 h_0 + 18 = 24$ and $\Delta h_2 = \Delta h_1 + 24 = 31$, we get $h_3 = h_2 + 31 = 41$. By Newton's forward formula,

$$h_n = 2\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 18\binom{n}{3} = 3n^3 - 6n^2 + 4n + 2.$$

Using $\sum_{k=0}^{n} \binom{k}{j} = \binom{n+1}{j+1}$,

$$\sum_{k=0}^{n} h_k = 2\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 18\binom{n+1}{4}.$$

3. (10 points) We want to tile a $2 \times n$ board using two types of tiles: (1) a 1×2 domino (which can be placed vertically or horizontally), and (2) a 2×2 square tile. Let h_n be the number of ways to tile the $2 \times n$ board.

(a) Find the recurrence relation for h_n and state the initial conditions h_1 and h_2.

Answer: $h_n = \underline{\ h_{n-1} + 2h_{n-2}\ }$, $h_1 = \underline{\ 1\ }$, $h_2 = \underline{\ 3\ }$.

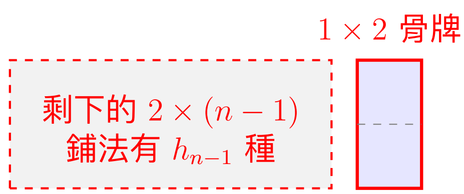
(b) Solve the recurrence relation to find an explicit formula for h_n.

Answer: $h_n = \underline{\ \frac{2}{3} \cdot 2^n + \frac{1}{3} \cdot (-1)^n\ }$

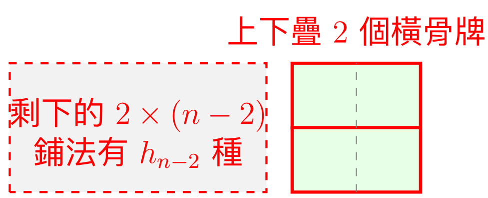
Solution：

(a) 為了填滿 $2 \times n$ 的棋盤，我們觀察「最右側」是如何被填滿的。總共有 3 種互斥的情況：

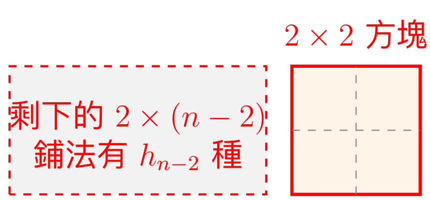
情況 1：最右側放置 1 個直立的 1×2 骨牌 $\implies$ 此情況的方法數：$1 \times h_{n-1} = \mathbf{h_{n-1}}$



情況 2：最右側放置 2 個橫躺的 1×2 骨牌 $\implies$ 此情況的方法數：$1 \times h_{n-2} = \mathbf{h_{n-2}}$



情況 3：最右側放置 1 個 2×2 方塊 $\implies$ 此情況的方法數：$1 \times h_{n-2} = \mathbf{h_{n-2}}$



將這三種互斥情況的方法數相加，即可得到遞迴式：

$$h_n = h_{n-1} + h_{n-2} + h_{n-2} \implies \mathbf{h_n = h_{n-1} + 2h_{n-2}}$$

Initial conditions (初始條件):

- $n = 1$: 棋盤為 2×1，只能放 1 個直立骨牌 $\implies h_1 = 1$
- $n = 2$: 棋盤為 2×2，可放 2 個直立、2 個橫躺、或 1 個方塊 $\implies h_2 = 3$

(b) Solving the Recurrence:
The characteristic equation is $r^2 - r - 2 = 0 \implies (r-2)(r+1) = 0$. The roots are $r_1 = 2, r_2 = -1$. The general solution is $h_n = c_1 2^n + c_2(-1)^n$. Using initial conditions: $h_1 = 2c_1 - c_2 = 1$ $h_2 = 4c_1 + c_2 = 3$ Adding the two equations gives $6c_1 = 4 \implies c_1 = \frac{2}{3}$. Substitute back to find $c_2 = \frac{1}{3}$. Thus, $h_n = \frac{2}{3} \cdot 2^n + \frac{1}{3} \cdot (-1)^n$.

4. (10 points) Solve the nonhomogeneous recurrence relation $h_n = 4h_{n-1} - 4h_{n-2} + 3^n$ with initial values $h_0 = 10$ and $h_1 = 35$.

(a) The form of the particular solution is $h_n^{(p)} = $ $\underline{\quad 9 \cdot 3^n \ \textbf{(or } 3^{n+2} \quad}$.

(b) The general solution is $h_n = $ $\underline{\quad 2^n + 3n \cdot 2^n + 3^{n+2} \quad}$.

Solution：

解法一：generating function （此解法可以不用寫 particular solution）

Let $G(x) = \sum_{n=0}^{\infty} h_n x^n$ be the generating function of the sequence h_n.
Rewrite the recurrence relation for $n \geq 2$:

$$h_n - 4h_{n-1} + 4h_{n-2} = 3^n$$

Multiply both sides by x^n and sum from $n = 2$ to ∞:

$$\sum_{n=2}^{\infty} h_n x^n - 4x \sum_{n=2}^{\infty} h_{n-1} x^{n-1} + 4x^2 \sum_{n=2}^{\infty} h_{n-2} x^{n-2} = \sum_{n=2}^{\infty} 3^n x^n$$

Express each term using $G(x)$, with $h_0 = 10$ and $h_1 = 35$:

$$\begin{aligned}
\sum_{n=2}^{\infty} h_n x^n &= G(x) - h_0 - h_1 x = G(x) - 10 - 35x \\
-4x \sum_{n=2}^{\infty} h_{n-1} x^{n-1} &= -4x(G(x) - h_0) = -4xG(x) + 40x \\
4x^2 \sum_{n=2}^{\infty} h_{n-2} x^{n-2} &= 4x^2 G(x) \\
\sum_{n=2}^{\infty} (3x)^n &= \frac{1}{1-3x} - 1 - 3x = \frac{1 - (1-3x) - 3x(1-3x)}{1-3x} = \frac{9x^2}{1-3x}
\end{aligned}$$

Substitute these back into the equation:

$$(G(x) - 10 - 35x) - 4xG(x) + 40x + 4x^2 G(x) = \frac{9x^2}{1-3x}$$

Factor out $G(x)$ and simplify:

$$G(x)(1 - 4x + 4x^2) - 10 + 5x = \frac{9x^2}{1-3x}$$

$$G(x)(1-2x)^2 = 10 - 5x + \frac{9x^2}{1-3x} = \frac{(10-5x)(1-3x) + 9x^2}{1-3x} = \frac{24x^2 - 35x + 10}{1-3x}$$

So, get the generating function and use partial fraction decomposition

$$G(x) = \frac{24x^2 - 35x + 10}{(1-2x)^2(1-3x)} = \frac{-2}{1-2x} + \frac{3}{(1-2x)^2} + \frac{9}{1-3x}$$

Now, expand each term into a power series using $\frac{1}{(1-ax)^2} = \sum_{n=0}^{\infty}(n+1)a^n x^n$:

$$G(x) = -2\sum_{n=0}^{\infty} 2^n x^n + 3\sum_{n=0}^{\infty}(n+1)2^n x^n + 9\sum_{n=0}^{\infty} 3^n x^n$$

Extract the coefficient of x^n to find h_n:

$$h_n = -2\cdot 2^n + 3(n+1)2^n + 9\cdot 3^n = 2^n + 3n\cdot 2^n + 3^{n+2}$$

Note: In the generating function approach, the fraction $\frac{9}{1-3x}$ directly produces the particular solution $h_n^{(p)} = 9\cdot 3^n = 3^{n+2}$, while the fractions with $(1-2x)$ in the denominator produce the homogeneous solution $h_n^{(h)} = (1+3n)2^n$.

解法二：公式解

Homogeneous Solution: The characteristic equation is $r^2 - 4r + 4 = 0 \implies (r-2)^2 = 0 \implies r = 2$ (double root). $h_n^{(h)} = c_1 2^n + c_2 n 2^n$.

Particular Solution: Since the non-homogeneous term is 3^n and 3 is not a characteristic root, guess $h_n^{(p)} = A\cdot 3^n$. Substitute this into the recurrence:

$$A\cdot 3^n = 4A\cdot 3^{n-1} - 4A\cdot 3^{n-2} + 3^n$$

Divide by 3^{n-2}:

$$9A = 12A - 4A + 9 \implies 9A = 8A + 9 \implies A = 9$$

So, $h_n^{(p)} = 9\cdot 3^n = 3^{n+2}$.

General Solution: $h_n = c_1 2^n + c_2 n 2^n + 9\cdot 3^n$. Use initial conditions:

- $h_0 = 10 \implies c_1 + 0 + 9 = 10 \implies c_1 = 1$
- $h_1 = 35 \implies 2c_1 + 2c_2 + 27 = 35 \implies 2(1) + 2c_2 + 27 = 35 \implies 2c_2 = 6 \implies c_2 = 3$

Thus, the final solution is $h_n = 2^n + 3n\cdot 2^n + 9\cdot 3^n$ (which can also be written as $2^n + 3n\cdot 2^n + 3^{n+2}$).

5. (10 points) Determine the number of integral solutions of the equation

$$x_1 + 5x_2 + x_3 + x_4 = n, \quad \text{that satisfy } 0 \le x_1 \le 4,\ 0 \le x_2,\ 4 \le x_3,\ -1 \le x_4$$

(a) The generating function $G(x) = \underline{\dfrac{x^3}{(1-x)^3}}$.

(b) The coefficient of x^n in $G(x)$, i.e. $h_n = \underline{\binom{n-1}{2} = \dfrac{(n-1)(n-2)}{2}}$.

Solution：

$$G(x) = \left(\frac{1-x^5}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{x^4}{1-x}\right)\left(\frac{x^{-1}}{1-x}\right) = \frac{x^3}{(1-x)^3}.$$

$$\because \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n, \quad \therefore [x^n]G(x) = \binom{(n-3)+2}{2} = \binom{n-1}{2}.$$

完整寫法為 $h_n = \binom{n-1}{2}$ if $n \ge 3$, $h_n = 0$ if $n < 3$；但 $\binom{n-1}{2}$ 在 $n < 3$ 時本來就是 0，故不必加條件。

6. (10 points) Let h_n denote the number of n-digit numbers with all digits at least 4 (i.e. digits $4, 5, 6, 7, 8, 9$), such that 5 occurs an even number of times, 7 and 8 each occur an odd number of times, 9 occurs at least once, and there is no restriction on the remaining digits (4 and 6). Determine the generating function $g(x)$ for the sequence $h_0, h_1, h_2, \ldots$ and then find a simple formula for h_n.

Answer: (a) The generating function $g(x) =$ $\frac{1}{8}(e^{6x} - e^{5x} - e^{4x} + e^{3x} - e^{2x} + e^x + 1 - e^{-x})$,

(b) The coefficient of $\frac{x^n}{n!}$ in $g(x)$, i.e. $h_n =$ $\begin{cases} \frac{1}{8}(6^n - 5^n - 4^n + 3^n - 2^n + 1 - (-1)^n) & \textbf{if } n \geq 1 \\ 0 & \textbf{if } n = 0 \end{cases}$.

Solution：

We use the exponential generating function (EGF) to solve this arrangement problem. The available digits are $\{4, 5, 6, 7, 8, 9\}$ (6 digits in total). We determine the EGF for each digit based on its restrictions:

- Digit 5 (even times): $\frac{e^x + e^{-x}}{2}$
- Digits 7 and 8 (each odd times): $\left(\frac{e^x - e^{-x}}{2}\right)^2$
- Digit 9 (at least once): $e^x - 1$
- Digits 4 and 6 (no restrictions): $(e^x)^2 = e^{2x}$

The total EGF $g(x)$ is the product of these functions:

$$\begin{aligned} g(x) &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)^2 (e^x - 1)e^{2x} \\ &= \frac{1}{8}(e^x + e^{-x})(e^{2x} - 2 + e^{-2x})(e^x - 1)e^{2x} \\ &= \frac{1}{8}(e^{6x} - e^{5x} - e^{4x} + e^{3x} - e^{2x} + e^x + 1 - e^{-x}) \end{aligned}$$

To find the formula for h_n, we extract the coefficient of $\frac{x^n}{n!}$ from $g(x)$. Using the expansion $e^{kx} = \sum_{n=0}^{\infty} k^n \frac{x^n}{n!}$, we have for $n \geq 1$:

$$h_n = \frac{1}{8}\left(6^n - 5^n - 4^n + 3^n - 2^n + 1^n + 0 - (-1)^n\right)$$

(Note that the constant term $1 = e^{0x}$ contributes 0^n, which is 0 for $n \geq 1$).

For $n = 0$:

$$h_0 = \frac{1}{8}(1 - 1 - 1 + 1 - 1 + 1 + 1 - 1) = 0$$

7. (8 points) The number of partitions of a set of n elements into k distinguishable(可區分的) boxes (some of which may be empty) is k^n. By counting in a different way, prove that

$$k^n = \binom{k}{1}1!S(n,1) + \binom{k}{2}2!S(n,2) + \ldots + \binom{k}{n}n!S(n,n)$$

(If $k > n$, define $S(n,k)$ to be 0.) Here $S(n,k)$ are the Stirling number of the second kind.

Solution：

Check Ch8 Theorem 8.2.5 and theorem 8.2.6.

8. (7 points) Let $s(n,k)$ be the Stirling numbers of the first kind. Find $s(5,3) =$ __35__ .

Solution：

解法一：由定義 Expand the falling factorial:

$$x(x-1)(x-2)(x-3)(x-4) = x^5 - 10x^4 + 35x^3 - 50x^2 + 24x.$$

The coefficient of x^3 is $s(5,3) = 35$.

解法二：用遞迴 $s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k)$。
邊界 $s(0,0) = 1,\ s(n,0) = 0\ (n \geq 1),\ s(n,k) = 0\ (k > n)$。逐列計算：

$n\backslash k$	1	2	3	4	5
1	1				
2	-1	1			
3	2	-3	1		
4	-6	11	-6	1	
5	24	-50	$\boxed{35}$	-10	1

例如 $s(5,3) = s(4,2) - 4\,s(4,3) = 11 - 4(-6) = 35$。故 $s(5,3) = 35$.

解法三：計算 5 元置換中恰含 3 個輪換的個數

恰 3 個輪環 ⇔ 輪環長度為「5 拆成 3 個部分」：型態 3+1+1 或 2+2+1。

$$\underbrace{\binom{5}{3}(3-1)!}_{3+1+1} + \underbrace{5 \cdot \tfrac{1}{2!}\binom{4}{2}}_{2+2+1} = 10 \cdot 2 + 5 \cdot 3 = 20 + 15 = 35.$$

故恰含 3 個輪環的 5 元置換有 35 個，即 $s(5,3) = 35$。

9. (5 points) Find the Ferrers diagram of the given partition $\lambda : 24 = 8 + 6 + 4 + 3 + 2 + 1$, and the determine the conjugate partition λ^*.

Answer: $\lambda^* =$ $24 = 6 + 6 + 4 + 3 + 2 + 2 + 1 + 1$

Solution：

Using the Ferrers diagram (費雷爾圖):

8	o	o	o	o	o	o	o	o
6	o	o	o	o	o	o		
4	o	o	o	o				
3	o	o	o					
2	o	o						
1	o							

Reading column by column from left to right.

Conjugate partition: $6 + 6 + 4 + 3 + 2 + 2 + 1 + 1$.

10. (10 points) Let p_n^s equal the number of self-conjugate partitions of n. Find p_{16}^s. *Hint:* By Theorem 8.3.2, $p_n^s = p_n^t$, where p_n^t is the number of partitions of n into distinct odd parts.

Answer: $p_{16}^s =$ 5

Solution：

Partitions of 16 into distinct odd parts, and their self-conjugate images:

p_{16}^t (distinct odd parts)	p_{16}^s (self-conjugate)
$15 + 1$	$8 + 2 + 1 + 1 + 1 + 1 + 1 + 1$
$13 + 3$	$7 + 3 + 2 + 1 + 1 + 1 + 1$
$11 + 5$	$6 + 4 + 2 + 2 + 1 + 1$
$9 + 7$	$5 + 5 + 2 + 2 + 2$
$7 + 5 + 3 + 1$	$4 + 4 + 4 + 4$

(3 個相異奇數之和為奇數，不可能等於偶數 16，故只有 2 個與 4 個部分的情形。) Hence $p_{16}^s = p_{16}^t = 5$.

11. (10 points) Let f_n is the n^{th} Fibonacci number. Prove that f_n is divisible by 3 if and only if n is divisible by 4.

Solution：

check Ch 7 problem 3

12. (5 points) Construct a linear homogeneous recurrence relation with constant coefficients for which $h_n = n^2$ is a solution (state the recurrence and a set of initial conditions).

Answer: $h_n = 3h_{n-1} - 3h_{n-2} + h_{n-3}$**, with** $h_0 = 0,\ h_1 = 1,\ h_2 = 4$.

Solution：

大家是否是沒看懂題目？把中文翻譯給你們：

構造一個具有常係數的線性齊次遞推關係，使得 $h_n = n^2$ 是一個解（寫出遞推關係式和一組初始條件）。

基本上可以從第七章的公式解去想就是了

13. (10 points) Let m and n be nonnegative integers with $n \geq m$. There are m + n people in line to get into a theater for which admission is 50 cents. Of the m + n people, n have a 50-cent piece and m have a $1 dollar bill. The box office opens with an empty cash register. Show that the number of ways the people can line up so that change is available when needed is

$$\frac{n-m+1}{n+1}\binom{m+n}{m}$$

Solution：

Ch 8 problem 5.

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
Points:	10	10	10	10	10	10	8	7	5	10	10	5	10	115
Score:														