BIJECTIVE PROOF PROBLEMS - SOLUTIONS

STOYAN DIMITROV[1], LUZ GRISALES[2], RODRIGO POSADA[2], MICHAEL SCHLEPPY[1]

This file contains solutions or references to solutions for more than 150 problems in the R.Stanley's list of bijective proof problems [3]. References to articles over a few of the unsolved problems in the list are also mentioned. At the end, we add some additional problems extending the list of nice problems seeking their bijective proofs.

1. Elementary Combinatorics

1. [**1**] Suppose you want to choose a subset. For each of the n elements, you have two choices: either you put it in your subset, or you don't;

More formally, let $A = \{a_1, a_2, \ldots, a_n\}$ be an n-element set. Consider an arbitrary subset $S \subset A$ and construct its characteristic vector $V_S = (v_1, v_2, \ldots, v_n)$ by the rule - $v_i = 1$ if $a_i \in S$ and 0 otherwise. On the other hand, each $0-1$ vector of length n corresponds to a unique subset of A. This is a bijection from V to 2^A, where V is the set of all $0-1$ vectors of length n. V has 2^n elements since we have to put either 0 or 1 (2 choices) to n different positions.

2. [**1**] A composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is uniquely determined by the set of the partial sums $\{\alpha_1, \alpha_1+\alpha_2, \ldots, \sum_{i=1}^{i=k-1} \alpha_i\}$, which is a subset of $[n-1]$. Here, note that the composition $\alpha = n$ corresponds to the empty subset of $\{1, 2, \ldots, n-1\}$. Conversely, for any subset of $[n-1]$, we have exactly one corresponding composition α. Just sort the set's elements in ascending order and take the differences between every two consecutive elements. The map is one-to-one and by problem 1, the number of subsets of $[n-1]$ is 2^{n-1}.

3. [**2**] Use the "dots and bars" model. The bar putted after dot i participates in 2^{n-2} partitions in total (choose 'put'/ 'don't put a bar' after the remaining $n-2$ dots among $1, \ldots, n-1$). Thus we have $(n-1).2^{n-2}$ parts ending before the n-th dot, in total. That way, the last part in every composition is missed. After we add it, i.e. after addition of 2^{n-1} parts, we get $(n-1).2^{n-2} + 2^{n-1} = (n+1).2^{n-2}$.

4. [**2-**] Assume you have determined to put or not a bar for each of the positions after the first $n-2$ dots. You can do this in 2^{n-2} ways. You get a composition with last part of length ≥ 2. Take the number of even parts in the received composition. If this number is even, keep the composition as it is. Otherwise, put a bar on the last possible position, i.e. after dot $n-1$. In fact, this way you divide the last part of length $l \geq 2$ to two parts

[1] Rutgers University New Brunswick
[2] Massachusetts Institute of Technology
E-mail address: emailToStoyan@gmail.com, mas991@scarletmail.rutgers.edu , luzg@mit.edu, rposada@mit.edu.

of lengths $l-1$ and 1. As a result, the parity of the last part is changed and the received composition will have even number of even parts.We actually get a bijection between the compositions with odd number of even parts and those with even number of even parts.
Example: $1+1+1+1 \to 1+1+2$
$3+1 \to 4$
$1+3 \to 1+2+1$
$2+2 \to 2+1+1$

5. [**2**] a) For every number a in $\{1,\dots,n\}$, we just need to choose the least index i_a of a set , s.t. $a \in S_{i_a}(i_a = 0$ if neither of the subsets contains $a)$ in order to determine the sets $S_1 \subseteq S_2 \subseteq \dots \subseteq S_k$. We see that for every a we have $k+1$ possibilities for the choice of this index, namely $0, 1, \dots, k$. Thus the answer is $(k+1)^n$.

b) The answer is $(k+1)^n$ again. Use a) and the fact that for every set of subsets with the inclusion $S_1 \subseteq S_2 \subseteq \dots \subseteq S_k$ we have a corresponding set of pairwise disjoint subsets $S_1, S_2 \setminus S_1, \dots S_k \setminus \cup_{i=1}^{i=k-1} S_i$. Another way to get the answer is to observe that for every number a in $\{1,\dots,n\}$ could belongs only to a set with index $i(S_i)$, $i = 1,\dots,k$ or to neither of those sets, i.e. $k+1$ cases for n different numbers.

c) Here, as in the previous parts of the problem, in order to determine such configuration, we have to choose one $0-1$ vector of length k for every number $i \in \{1,\dots,n\}$. The additional condition is that none of these vectors could be consisted of 1s entirely. Therefore, the answer is $(2^k-1)^n$.

6. [**1**] Interpretation of the LHS - Choose k different elements out of n in $\binom{n}{k}$ ways and order them in $k!$ ways. Instead(interpr. of the RHS) we can choose the element ordered as first in n ways, then the second in the order in $n-1$ ways, etc. , the kth in the order in $n-k+1$ ways.

7. [**1+**] Assume we have distributed the paranthesis in $(x+y)^n = \underbrace{(x+y)(x+y)\dots(x+y)}_{n}$.
Every summand of the result will be of the kind $x^i y^{n-i}$ for some $i = 0, 1, \dots, n$. Let's fix some $0 \le k \le n$. How many of the summands are going to be of the kind $x^k y^{n-k}$? In order to have such summand we had to choose the x term out of exactly k paranthesis which could be done in $\binom{n}{k}$ different ways.

8. [**1**] Consider the problem in the general case where you have to reach the point (n,m). The path consist of exactly n steps to the right and m steps up. To determine the path it suffices to choose the moments where we make the steps to the right. This can be done in $\binom{n+m}{n}$. The latter quantitiy is equal to $\binom{n+m}{m}$.

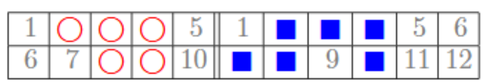
9. [**1**] Consider the LHS. For every S - an n-subset of $[2n-1]$, consider the 2 n-subsets of $[2n]$ - S and $\{2n\} \cup ([2n-1] \setminus S)$.The map is bijective.

10. [**1+**] Take the elements of $[n]$. Take the element 1(we know that $n \ge 1$). For every subset S of $[n]$ with k elements, where k is even, we can get a unique subset with odd number of elements as follows: if $1 \in S$ remove it, otherwise add it.

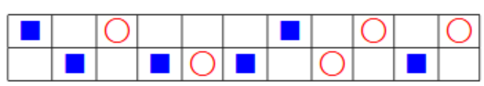
11. [**1+**] Let us choose n elements from the set $\{1, 2, \ldots, x, x+1, \ldots, x+y\}$. Some of these n elements have to be among $1, 2, \ldots, x$. Let the number of those be k. The remaining $n-k$ elements have to be among $x+1, \ldots x+y$. There exists $\binom{x}{k}\binom{y}{n-k}$ such configurations and k could be between 0 and n. This is the so-called Vandermonde identity.

12. [**2-**] Consider a choice of n elements out of $x+n+1$ elements. Among the last $n+1$ there must be at least one unchosen element. Denote the unchosen element with the biggest number with $x+k+1$. We know that the $n-k = (x+n+1)-(x+k+1)$ elements having bigger numbers are selected, thus in order to determine the choice of the n elements completely, we have to select the rest $n-(n-k) = k$ elements from the remaining $x+k$ elements. This reasoning shows that the given equality is a partition of all possible choices of n elements out of $x+n+1$ elements by the number of the largest unchosen element.

13. [**3**] A combinatorial proof from 2013 can be found in the paper "New developments of an old identity" by Duarte and de Oliveira.
Instead of the LHS, they first consider the set of slots of two rectangles, of size $2 \times i$ and $2 \times j$, respectively, ordered row by row and joined in a $(2 \times n)-$rectangle R$(i+j=n)$, such that exactly n of the $2n$ slots are colored - those in the left rectangle with one color and those in the right triangle with another color(see the figure below, containing an example). They call these $2 \times n$ rectangles - 'ordered n-configurations'. If we remove the condition to use one color on the left and the other on the right, then we call these $2 \times n$ rectangles just 'n-configurations'.

$R =$ 

If two of the slots in a column are colored, they call this column - tower. For the example above, we have exactly 4 towers. We may easily see that the LHS is the number of ordered n-configurations, while the total number of n-configurations without towers is exactly 4^n(for each of the n columns one have 2 choices for the slot being selected and 2 choices for the color of this slot). The paper provides a bijection between these 2 sets.



n-configuration without towers

14. [**3-**] One old reference is "Identities in combinatorics, I: on sorting two ordered sets" by George Andrews. You can also look at this math.stackexchange topic.

15. [**3-**] Note that both sides of the equality represent a polynomial of degree n in x. Therefore it is enough to prove that the equality holds for the positive integer values of x to show that the two polynomials are identical. Now, assuming x is a positive integer, consider the following situation:
A company wants to hire n people. Among the $2n$ applicants, there are exactly n applicants who applied for a specific role, and exactly n applicants who are willing to work in any of

the x available roles. We should note that the company can hire many people (or none) for each role.
We want to determine T, the number ways we can choose n people to hire and assign a role to the ones who didn't specify one.
If the company hires k applicants who didn't specify a role, the number of possible outcomes is $\binom{n}{k}\binom{n}{n-k}x^k = \binom{n}{k}^2 x^k$.
Adding this over all the possible values of k, we get:

$$T = \sum_{k=0}^{n} \binom{n}{k}^2 x^k.$$

Another less intuitive way to count it is by setting aside a special role (with $x-1$ non-special roles remaining). Then, in the case we assign exactly j people to a non-special role, the number of possible outcomes is $\binom{n}{j}\binom{2n-j}{n-j}(x-1)^j = \binom{n}{j}\binom{2n-j}{n}(x-1)^j$. This is because there are $\binom{n}{j}(x-1)^j$ possible ways to choose j people without a specific role and assign them a non-special one. Once chosen, there are $\binom{2n-j}{n-j}$ ways to choose the remaining $n-j$ people and whoever didn't specify a role, will be assigned to the special one.
Adding this over all the possible values of j, we get:

$$T = \sum_{j=0}^{n} \binom{n}{j}\binom{2n-j}{n}(x-1)^j.$$

See also, R.Chen, C. Reidys, "A Combinatorial Identity Concerning Plane Colored Trees and its Applications", J. Integer Seq. 20.3 (2017): 17-3.

16. [**3-**] We have that $\binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{i}$ is the number of triples (α, β, γ), where:

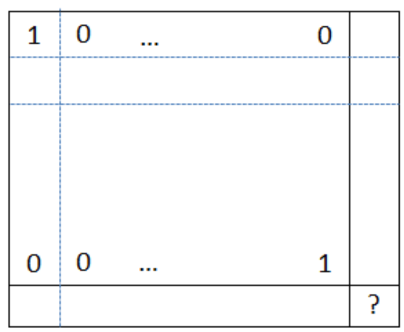
(i) α is a sequence of $i+j+2$ letters a and b beginning with a and ending with b, with $i+1$ a's (and hence $j+1$ b's),
(ii) $\beta = (\beta_1, \dots, \beta_{j+1})$ is a sequence of $j+1$ positive integers with sum $j+k+1$,
(iii) $\gamma = (\gamma_1, \dots, \gamma_{i+1})$ is a sequence of $i+1$ positive integers with sum $k+i+1$.

Replace the r-th a in α by the word $c^{\gamma_r}d$, and replace the r-th b in α by the word $d^{\beta_r}c$. In this way we obtain a word δ in c, d of length $2n+4$ with $n+2$ c's and $n+2$ d's. This word begins with c and ends with $d(dc)^m$ for some $m \geq 1$ (To see this it helps to consider the chain of 1's at the end of β and separate in cases). Remove the prefix c and suffix $d(dc)^m$ from δ to obtain a word ϵ of length $2(n-m+1)$ with $n-m+1$ c's and $n-m+1$ d's. The map $(\alpha, \beta, \gamma) \longrightarrow \epsilon$ yields a bijective proof by counting the same set in two ways ($r = n-m+1$). This argument is due to Roman Travkin, from Stanley's book [5]

Example. Let $n = 8$, $i = 2$, $j = k = 3$, $\alpha = abbaabb$, $\beta = (2, 3, 1, 1)$, $\gamma = (2, 3, 1)$.
Then $\delta = (c^2d)(d^2c)(d^3c)(c^3d)(cd)(dc)(dc)$, so $\epsilon = cd^3cd^3c^4dc$ and $m = 2$.

17. [*] An elementary proof using recursion and induction was given in Mikic, J., A proof of Dixonâs identity, J.Integer Sequences, 19, 2016. His argument was further generalized to q-analogues in Guo, V.J., A new proof of the q-Dixon identity. Czechoslovak Mathematical Journal, 68(2), pp.577-580, 2018.

19. [**2-**] Denote our matrix with A. The answer is $2^{(m-1)\times(n-1)}$. First notice that m and n have to be of the same parity. Otherwise, we'll get that the sum of the matrix' elements by rows and those by columns have different parity. Let's fill all of the entries of the matrix with 0s and 1s at random, except those in the last row and in the last column(see Figure). In fact we are filling at random a $(m-1)\times(n-1)$ matrix A' and this could be done in $2^{(m-1).(n-1)}$ ways. Each row and column of A have odd number of 1s thus all of the remaining entries' values are strongly determined except the entry in the right-down corner of A(marked with '?'). Indeed, if the current number of 1s is odd, we can write 0 in the last entry and 1 otherwise. We'll call the values in those last entries "generated". Notice that we have $(m-1)+(n-1)$ generated values in total. Let's show that we have exactly one possible value for "?". If the sum of the randomly chosen elements of A' is S and S is WLOG even, then the last row of A and the last column of A will be filled with even number of generated 0s. Since $m-1$ and $n-1$ have to be of the same parity too, the number of generated 1s in the last row and column must also be of the same parity. If this number is odd, the "?" have to be 0 and 1 otherwise, which shows that we always have an appropriate value for the "?" and it's uniquely determined after the random filling of A'. The same reasoning is valid when S is odd.



20. **a)**[**1-**] a_1 could be chosen in k ways, while for every a_i for $i>1$ one would have $k-1$ possible choices(every possible value except the previous one).

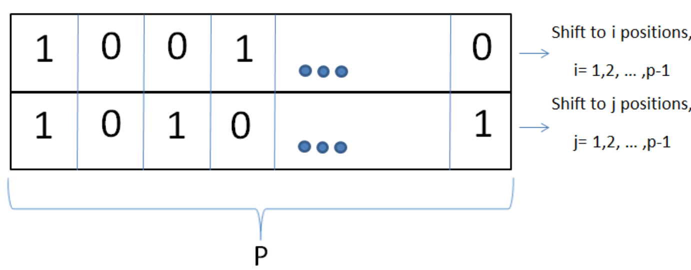
b)[**2+**] Let's show first that

$$g_k(n-1)+g_k(n)=k(k-1)^{n-1} \tag{1}$$

We'll call the $g_k(n)$ sequences of that property - (k,n)- circular and those in a) - (k,n)-non-circular. The RHS is the number of all (k,n)-non-circular sequnces. Thus it suffices to prove that the number of (k,n)-non-circular sequences such that $a_1=a_n$ is $g_k(n-1)$. Indeed, we'll have that $a_{n-1}\neq a_n=a_1$. Therefore, for every such (k,n)-non-circular sequences we have one $(k,n-1)$- circular, namely $a_1,a_2,\ldots,a_{n-1}$. Conversely, from every $(k,n-1)$-circular seq. $a_1,a_2,\ldots,a_{n-1}$ we can make the (k,n)-non-circular seq. $a_1,a_2,\ldots,a_{n-1},a_1$. The correspondence is one to one. Now, using induction and the fact that $g_k(1)=0$ we can prove 1 easily.

21. [**2-**] a^p is the number of the different vectors with p entries, s.t. each entry is among the numbers $1\ldots a$. Take these vectors and exclude those of them that have all components equal (the number of these vectors is a). Now, because p is prime, we can observe that after taking an arbitrary vector among the remaining a^p-a vectors and after we shift this vector cyclically with k positions ($k = 1, \dots p-1$), we must always obtain a different vector. Indeed, assume the opposite, i.e., for some vector $x = (x_1, x_2, \dots, x_p) \in [a]^p$ and certain $k \in \{1, \dots p-1\}$, $x = (x_{k+1}, x_{k+2}, \dots x_{k+p})$, where the indices are taken by module p. Notice that this would imply also that $x = (x_{2k+1}, x_{2k+2}, \dots x_{2k+p})$, $x = (x_{3k+1}, x_{3k+2}, \dots x_{3k+p})$, etc. But, take for instance the first entry having value x_1. We will have that $x_1 = x_{i.k+1}$, for all $i = 1, 2, \dots, p$, where $i.k + 1$ is in $\mathbb{Z}_p$. From p- prime, follows that $i_1.k + 1 \neq i_2.k + 1$ for $i_1 \neq i_2$ since all $k, i_1, i_2 < p$. The latter means that all of the elements of x must have the same value, which is not true (we excluded those vectors). Thus for each vector we obtain another $p-1$ different vectors which form altogether a class of p different vectors. Therefore our set of $a^p - a$ vectors could be divided that way in groups of p vectors and the claim follows.

22. a)[2] Take a set of $2p$ elements A. Consider the set of all $p-$subsets of A. Represent one such subset by a $2 \times p$ table of 0s and 1s, containing exactly p 1s (see the figure). Remove the trivial choices, i.e., those 2 choices having all 1s in the first row or in the second row. Thus, all elements of the remaining set of choices have both 0s and 1s in each of the two rows. Therefore, if we shift the first row by i positions, $i = 0, 1, 2, \dots p-1$, we would always obtain a different $0-1$ vector. The latter follows as a special case of problem 20 for $a = 2$. The same could be said for the second row. Now, we see that each choice of $p - subset$ belongs to a class of $p \times p$ choices obtained by shifting the first and the second row of this choice to i and j positions respectively($i, j = 0, 1, \dots p-1$). Every such class is closed thus $\binom{2p}{p} - 2$ could be divided by p^2(we removed the two trivial choices).



b)[*] In 1819, inspired by Wilson's Theorem, which asserts that if p is prime

$$(p-1)! \equiv -1 \pmod{p}, \tag{2}$$

Charles Baggage [1] proved for odd primes p that

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}. \tag{3}$$

In 1862, the Reverend Wolstenholme [8] improved Babbage's Theorem by proving for primes $p > 3$

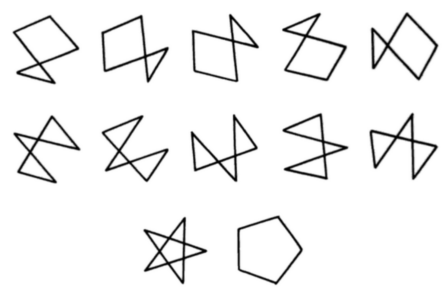
$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}. \tag{4}$$

There is one piece of evidence that the p^2 congruence (Babbage's theorem) is entirely combinatorial, but the extension to p^3 (Wolstenholme) is essentially algebraic. First, that Wolstenholme's theorem doesn't hold for the primes $p = 2, 3$.

.

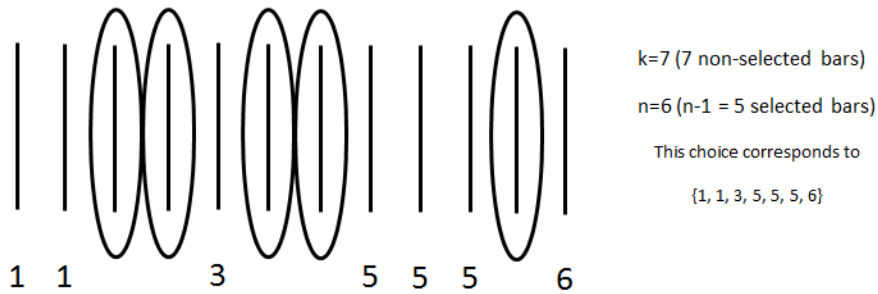
23. [**2-**] This is a proof from the book of G. Andrews "Number Theory". It is a widely known proof which was first found by J. Petersen, Tidsskrift for Mathematik (3), 2, 1872, 64-5. according to Dickson's History of the Theory of Numbers.

The case $p = 2$ is directly checked thus we can assume that p is odd prime. Take p points on a circle such that they divide it into p equal arcs. If we connect these points in some order, how many different polygons can be formed(crossing of edges is allowed)? One would say $p!$ as this is the number of orderings of the points, but we every polygon corresponds to $2p$ orderings since the points are on a circle(thus we can start from each of the p vertices of the polygon) and being in the first vertex, we can continue with either of the two vertices connected with it. Now, let's observe that from the $p!/2p = (p-1)!/2$ polygons with p vertices, there are exactly $(p-1)/2$ which stay the same after we move each vertex to the next one, i.e. after rotation to an angle $\frac{2\pi}{p}$. Indeed, for those polygons, the distance between any two vertices have to be the same, since we could perform arbitrary number of rotations and the result is always the same polygon. We have $(p-1)/2$ different possible distances when $1, 2, \ldots, (p-1)/2$ dots are skipped respectively. Now, using the same observation as in the previous problems, we see that the remaining $(p-1)!/2 - (p-1)/2 = \frac{(p-1)!-p+1}{2}$ polygons can be naturally divided into sets of p elements obtained after rotation to angles $\frac{2k\pi}{p}$, $k = 0, 1, \ldots p-1$. Since p is prime there are no two polygons which are the same among those received after the rotations. Then $(p-1)! - p + 1$ divides by p and we're done. Below are given the 12th polygons when $p = 5$, taken from the book of G.Andrews.



24. [**1**] The answer is $\prod_{i=1}^{k} v_i + 1$ since every subset S is determined by its elements and for each $i = 1, \ldots, k$, S could have one among $0, 1, \ldots v_i$ copies of i.

25. [**2**] For simplicity take the n-set to be $[n]$ and use that $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$. Let's take some set of $n+k-1$ identical objects e.g. vertical bars. Consider some subset of size $n-1$ of it. The remaining bars are exactly k in number and they are naturally divided into n groups corresponding to the numbers from 1 to n. How? Write 1s for all the bars before the first chosen bar. Write 2s for all the unchosen bar between the first and the second selected bars, and so forth. At the end, write n for each non-selected bar after the last selected bar(see the figure). Conversely, given a submultiset given by the vector $(k_1, k_2, \ldots k_n)$, for which k_i is the number of elements equal to the$i$-th in the submultiset($0 \le k_i \le k$, $\forall i = 1, \ldots, n$), we have a direct way to obtain $n-1$ chosen bars out of $n+k-1$. Put k_1 unchosen bars then one chosen, k_2 unchosen bars then one chosen and so forth. Don't put a chosen bar at the very end. The two defined maps are apparently injective.



26. [**2-**] The same as finding the number of submultisets of $[k]$ of size n after interpreting x_i as the number of elements i in the submultiset. In other words we got the same problem as the previous one with n and k switched. Thus the answer is $\binom{n+k-1}{n}$.

Note: We got one popular variation of the problem when $x_i \geq 1$, but not 0. Then we can reduce the problem to our initial by taking $x_i' = x_i - 1$ and by solving for the vector x'.

27. [*] Look at A.Haupt, "Combinatorial Proof of Selberg's Integral Formula", arXiv preprint:2005.08731 (2020).

28. [*] Look at section 4 of H. Bidkhori, S. Kishore, "Counting the spanning trees of a directed line graph" (2009). See also [2], page 159, Corollary 10.11.

30. [**1**] Clearly, $f(1) = 1 = F_2$ and $f(2) = 2 = F_3$. Consider the $f(n)$ compositions of $n \geq 3$ with parts 1 and 2. The number of them having last part 1 is $f(n-1)$ and those having last part 2 is $f(n-2)$ since after removing this last part, we obtain another compositions with parts 1 and 2 for $n-1$ and $n-2$ respectively. Also, having any of the $f(n-2)$(or the $f(n-1)$) compositions to 1s and 2s of n-2 (or n-1), we get such composition for n by adding 2 (or 1) at the end.

31. [**2-**] Denote the number of compositions of n with all parts > 1 with $g(n)$. Trivially, $g(2) = g(3) = 1$. Consider all the $g(n)$ compositions for some $n > 3$. If the last part is 2 for some of them and we remove this part, we get a composition of $n-2$ having all parts > 1. Now, consider the case when the last part is ≥ 3. If we remove 1 from this last part, we obtain partition with all parts > 1 of $n-1$. On the other hand, we can add one part of size 2 to the end of each composition among those $g(n-2)$ or 1 to the last part of each such composition of $n-1$ to obtain one of the compositions in $g(n)$. Since the only options are to have last part 2 or last part ≥ 3 and these two subsets of compositions does not intersect, we can conclude that $g(n) = g(n-1) + g(n-2)$.

32. [**2-**] Denote the number of described compositions of n with $h(n)$. We have $h(1) = 1 = F(1)$ and $h(2) = 1 = F(2)$. Use similar observations to those for the previous problem. Given a composition with odd parts, we could have either last part 1 or otherwise we can remove 2 from this last part.

33. [**1+**] Let the number of those subsets be denoted with $g(n)$. Check that $g(1) = 2^1$ and $g(2) = 2^2 - 1 = 3$. Consider those $S \subseteq [n]$ containing n and those not containing n. For the former case, we know that $n-1 \notin S$ and $S \setminus n$ does not contain two consecutive integers. The set $S \setminus n$ could be any such subset of $[n-2]$. Thus we get $g(n-2)$ subsets of $[n]$ containing n and withouth two consecutive integers in them. For the case $n \notin S$, we know that $S \subset [n-1]$ without having two cons.integers in it and that each such S is a subsets ot $[n]$. Therefore, we obtain $g(n) = g(n-1) + g(n-2)$ for $n \geq 3$ and given the initial conditions we get that $g(n) = F_{n+2}$.

34. [**2-**] Denote this number by g(n) and call the sequences sufficing this property "good". If $\epsilon_1 = 0$, then $(\epsilon_2, \dots \epsilon_n)$ is also "good". In addition, for any "good" $(\epsilon_2, \dots, \epsilon_n)$ of length $n-1$, we can add one 0 in front to make a "good" sequence starting with a 0 of length n. Thus there are $g(n-1)$ such sequences. On the other hand, if $\epsilon_1 = 1$, then ϵ_2 have to be 1, too and the rest is a "good" sequence of length $n-2$(and this sequence could be any of those $g(n-2)$). Thus we have the relation $g(n) = g(n-1) + g(n-2)$ again and it remains to check that $g(1) = 2$ and $g(2) = 3$ to obtain that the answer is F_{n+2}.

35. [**2+**] See solution of p.35 after chapter 1 in EC2. As pointed there, the proof of this problem and the next two problems are due to Ira Gessel.
The first step is to observe that the LHS is the number of ways of inserting at most one bar in each of the $n-1$ spaces separating a line of n dots, and then selecting(circling) one dot in each compartment. The secont step is to see that if you put some number A for each uncircled dot and some other number B for each circled dot or bar, you get a representation of this configuration enough to recovers it. Indeed, if you take the Bs, you know that starting from the beggining, circled dots and bars are alternating. In addition, if we choose $A = 2$ and $B = 1$, we'll get a constant sum for our representation which is $2n-1$. Conversely, given any composition of $2n-1$ with 1s and 2s only, we know that the number of 1s have to be odd, so we'll have a corresponding "dots and bars" representation with selected dot in each compartment. By problem 30 we know that the number of those compositions is F_{2n}.

36. [**3-**] Given a composition $(a_1, a_2, \dots, a_k)$ of n, replace each part a_i with a composition α_i of $2a_i$ into parts 1 and 2, such that α_i begins with a 1, ends in a 2, and for all j the $2j$-th 1 in α is followed by a 1, unless this $2j$-th 1 is the last 1 in α. For instance, the part $a_i = 4$ can be replaced by any of the seven compositions $1111112, 111122, 111212, 11222, 121112, 12122, 12212$. Now, let's check the following two facts that suffices to get a proof:

i) every composition of $2n$ into parts 1 and 2, beginning with 1 and ending with 2, occurs exactly once by applying this procedure to all compositions of n. Indeed, assume we have such composition. One will be able to divide it into segments representing the different $2\alpha_i$-s forming a corresponding composition of n. The latter is true, because it is enough to look for those 2 followed by 1 and having even number of 1s in front of them. All of these 2s and only they are ends of a segments representing a part. For example, if we're given the $1-2$ composition 1111112111212122, we can see that it corresponds to $1111112 \mid 111212 \mid 12122$, which is one of the possible representations for the partition of 12 - (4,4,4).

ii) the number of compositions that can replace a_i is $2^{a_i-1}-1$. Well, here is a way that we can create all such compositions. Write 1 at the beginning, 2 at the end and choose a_i-1 times to write "11" or "2" for the middle portion(you can do this in 2^{a_i-1} ways). But, the total sum became $2n+1$. It remains to remove the last 1. However, this could be done if you have at least one 1. To gurantee this, just do not consider the choice of a_i-1 2s for the middle portion. Now, the number of ways you can perform the described steps is $2^{a_i-1}-1$.

It follows again from problem 30 that the answer is F_{2n-2}.

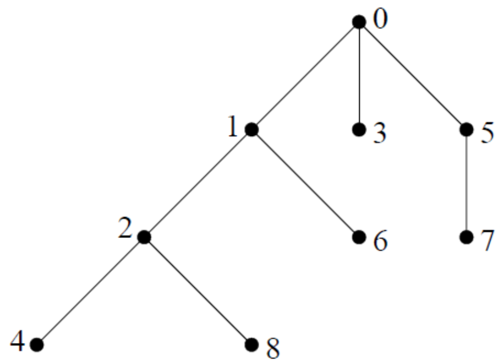
37. [**2**] Given a composition $(a_1, a_2, \dots a_k)$ of n, replace each 1 with either 2 or $1, 1$, and replace each $j>1$ with $1, 2, \dots, 2, 1$ where there are $j-1$ 2s. Every composition of $2n$ with parts 1 and 2 is obtained in this way. Indeed, assume that some of these compositions is given. Start from the beginning. If the first number is 2 - put the end of the part after it. If it's 1 - put the end of the part immediately after the next 1. It follows again from problem 30 that the answer is F_{2n+1}.

[40. [2-]] We'll count the number of compositions of $n+1$ with parts 1 and 2 that have at least one 2. Since there is only one composition of $n+1$ that doesn't have 2's, by the problem 30 there are $F_{n+2}-1$ such compositions. Now, let's count all the compositions of $n+1$ such that there are i ones before the first 2 appears. For this, $i \in \{0, \cdots, n-1\}$. The sequence of values after the first 2 must be a composition of $n+1-(i+2)$ so there are F_{n-i} possibilities. Adding this over all the possible values of i, we get the desired equality:

$$F_{n+2}-1 = \sum_{0}^{n-1} F_{n-i} = F_1 + F_2 + \cdots + F_n$$

41. [**2**] For every number $a \in \{1, \dots, n\}$ and every $i \in \{1, \dots, k\}$ write $\epsilon_i^a = 1$, if $a \in S_i$ and 0 otherwise. You'll get that for every fixed $a \in \{1, \dots, n\}$, the sequence $\{\epsilon_i^a\}_{i=1}^{i=k}$ suffices the condition $\epsilon_1^a \le \epsilon_2^a \ge \epsilon_3^a \le \epsilon_4^a \ge \epsilon_5^a \le \dots$. Using problem 34, we conclude that the answer is F_{k+2}^n.

42. [**1**] See solution of problem 63 a) after chapter 1 in EC2, as well as Proposition 1.5.5 there. Denote the biggest envelope with 1, the 2nd biggest with 2 and so on. Every envelope(together with the envelopes it contains) that is not in any other envelope corresponds naturally to an increasing labeled tree. We can combine these trees in one tree by making the roots successors of a common root labeled with 0. This way we showed a bijection between the arrangments of n envelopes and all the increasing labeled trees with labels $0, 1, \dots n$. Let's show now a bijection between those increasing trees and the permutations of $1 \dots n$. Given a permutation $w = w_1 w_2 \dots w_n \in \mathbb{G}_n$, construct a tree $T'(w)$ with vertices $0, 1, \dots, n$ by defining vertex i to be the successor of the rightmost element j of w which precedes i and which is less than i. If there is no such element j, then let i be the successor of the root 0. See the picture below, taken from EC2. Consider this example carefully to understand why the map is one-to-one.



The unordered increasing tree $T'(57316284)$

43. [**2+**] This problem was proposed as a question for IMO 2002. Let's call the described sequences "good". So, $f(n)$ is the cardinality of the set of good sequences of length n, denoted by G_n. Let's define a map $\phi : G_n \to \mathcal{G}_n$ as follows: $\phi(u) = a_1a_2 \ldots a_n$, where a_i is the position of the i-th leftmost largest integer of u. In other words, we sort the numbers in u in decreasing order and we simply list their positions in u such that if ties occur, we write the positions of these equal numbers in order, from the smallest to the biggest(as they occur from left to the right). Examples:

$$2132 \longmapsto 3142, 213421 \longmapsto 431526$$

You can show that this is a bijection(or look at "Good sequences, bijections and permutations" by Igor Kortchemski, 2005). In fact, knowing that for each $k > 1$, $k-1$ occurs before the last occurence of k, makes ϕ^{-1} one-to-one.

44. [**2-**] Consider the pairs (i, p) where p is an $[n]-$permutation and i is a fixed point in p. One can construct a bijection between this set of pairs and the set of permutations of $[n]$, by making a permutation of $[n]$ that contains i at the beginning and then take the remaining elements in the order they appear in p.

Formally, if $p = p_1 \ldots p_i \ldots p_n$ and i is a fixed point, i.e., $p_i = i$, then let

$$g((i, p)) = ip_1 \ldots p_{i-1}p_{i+1} \ldots p_n$$

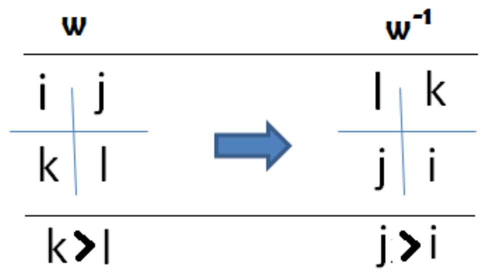
. One can easily describe the inverse map

$$g^{-1}(p) = p_2 \ldots p_{i-1}ip_{i+1} \ldots p_n,$$

where $p_i = i$. Therefore, g is bijective and the total number of fixed points is $n!$.

45. [**2**] Consider the RHS. Every permutation w of $1 \ldots n$ can be constructed consecutively as follows. First determine whether 2 is after or before 1. If it's after we don't have an inversion and if it's before we have exactly 1 inversion with bigger element 2. This choice corresponds to the first parenthesis of the RHS. Similarly, 3 could participate as a bigger element in $0, 1$ or 2 inversions respectively and this choice corresponds to the second parenthesis.

46. [**1**] Let $w = a_1a_2\dots a_n$. Take arbitrary inversion at positions i and j, i.e. $a_i > a_j$. But for this inversion, we have the corresponding inversion $j > i$ in w^{-1} since $w^{-1}(a_j) = j$ and $w^{-1}(a_i) = i$.



47. [**2-**] The answer is 2^{n-1}. Here is a way to construct all such permutations. Start from the position where just the element 1 is already placed. 2 can be placed either on the left or on the right of 1. Then 3 could be either to the left of both 1 and 2 or the right of them, but not between them. The same way, we always have two choices for the number i - being before all of the smaller numbers or after all of them. We have to make this choice for the $n-1$ numbers $2, 3, \dots, n$ in order to construct such permutation and all choices results in different permutations. Conversely, given a permutation with the given property, one can easily determine whether i is to the left or to the right of all previous numbers, for all $i = 2, 3, \dots, n$.

48. [**2-**] The union A_i of $a_1, a_2, \dots, a_i$ always form an interval for every $i = 1, 2, \dots, n-1$. Thus after choosing a_{i+1} we always extend this interval with one, either to the left or to the right. This means that we always have two possible choices unless 1 or n are already reached. After this moment, the permutation is uniquely determined(we can take only one direction of extension after that). The latter means that if $a_1 = k$, we have $\binom{n-1}{k-1}$ sequences(permutations) starting with k. Indeed, the remaining number of elements is $n-1$ and the permutation is determined by the positions of the elements $k-1, k-2, \dots, 1$ (the moments of extensions to the left). After summation for k we get $\sum_{k=1}^{k=n} \binom{n-1}{k-1} = 2^{n-1}$.

49. [**2**] $\frac{n!}{k!} = \frac{n!}{k!(n-k)!}.(n-k)! = \binom{n}{k}.(n-k)!$. The latter is true for $k = 0, \dots, n$, so each term of this kind corresponds to the number of ways to choose k fixed points and to determine the rest of a permutation. Thus, if a permutation w has l fixed points ($l = 0, 1, \dots, n$), it is counted $\binom{l}{1} + \binom{l}{3} + \dots$ on the left and $\binom{l}{0} + \binom{l}{2} + \dots$ on the right. As we know, these 2 quantities are equal if $l \geq 1$(see p.10). When $l = 0$, i.e. derangement, each such permutation is counted $\binom{0}{0} = 1$ time on the right and this is compensated by the term $D(n)$ on the left.

50. [**1**] Consider a derangement σ. Let $\sigma(k) = n$ ($k \neq n$) for some k in $\{1, 2, \dots, n-1\}$. The number of derangements σ for which $\sigma(n) \neq k$ is $D(n-1)$ since we can swap $\sigma(k)$ and $\sigma(n)$ and ignore the last position n. Conversely, for each of the $D(n-1)$ derangements of $1 \dots n-1$, we can add the fixed point n and swap it with any of the first $n-1$ positions. Similarly, the number of derangements σ for which $\sigma(n) = k$ is $D(n-2)$, because after swaping we can ignore both position - n and k.

51. [**3**] The solution below follows the paper of Fanja Rakotondrajao, "k-fixed-points-permutations", 2006. Another more recent (2017) and shorter paper is the one of Arthur Benjamin titled: "A bijective proof of a derangement recurrence"

Let's define the derangement $\Delta_n = (1\ 2)(3\ 4)\dots(n-1\ n)$ if the integer n is even and the sets

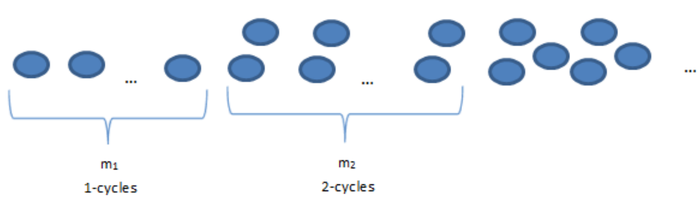
- $E_n = \{\Delta_n\}$ if the integer n is even and $E_n = \emptyset$ otherwise.
- $F_n = \{(n, \Delta_{n-1})\}$ if the integer n is odd and $F_n = \emptyset$ otherwise.

Let $\tau : [n]\times(D_{n-1}\setminus F_n) \to D_n\setminus E_n$ be the map which associates to a pair (i, δ) a permutation $\delta' = \tau((i, \delta))$ defined as below:

(1) If $i < n$, the permutation δ' is obtained by inserting n in the cycle which contains i just after i.
(2) If $i = n$, then let p be the smallest integer, s.t. $(1\ 2)(3\ 4)\dots(2p-1\ 2p)$ are cycles of the permutation δ and the transposition $(2p+1\ 2p+2)$ is not. Consider 2 cases here:
  (a) if $\delta(2p+1) = 2p+2$, then the permutation δ' is obtained from δ by removing $2p+1$ from the cycle which contains it, and then creating the new cycle $(2p+1\ n)$.
  (b) if $\delta(2p+1) \neq 2p+2$, then we have to distinguish the following 2 cases:
    (i) If the length of the cycle of $2p+1$ is 2, then obtain δ' from δ by removing this cycle $(2p+1\ \delta(2p+1))$, and then inserting the integer $2p+1$ in the cycle which contains $2p+2$, right before it. Create also the new cycle $(\delta(2p+1)\ n)$.
    (ii) If the length of the cycle of $2p+1$ is more than 2, then obtain δ' from δ by removing $\delta(2p+1)$ and then by creating the new cycle $(\delta(2p+1)\ n)$

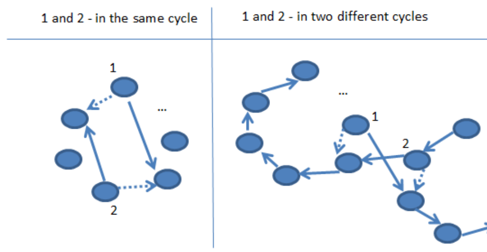
You may try to verify that this map is bijective(see the cited paper for a proof).

52. [**2**] First, fix any of the $n!$ orderings of the n elements of the given cycle structure(shown on the picture below). For each $i = 1, \dots, n$, we can exchange the places of the m_i cycles of length i and get the same permutation. This can be done in $m_i!$ ways. In addition, we can rotate the elements in each cycle of length i (which is the same as choosing a first element in each cycle) in i different ways and we'll get the same permutation. Therefore, each permutation with the given structure corresponds to $1^{m_1}m_1!2^{m_2}m_2!\dots n^{m_n}m_n!$ different initial orderings and the number of initial orderings is $n!$.



53. [**1+**] This is indeed a special case of p.52 when we only have cycles of length 2. Each element must be putted with another element in a cycle. The "spouse" of the first element can be chosen in $2n-1$ ways. Then, choose another of the remaining elements and select its "spouse" from one of the remaining $2n-3$ elements. Continue that way.

55. [**2-**] We shall prove that the number of permutations with even number of even cycles equals the number of permutations with odd number of even cycles. $n \geq 2$ so we can define the map $\tau : \sigma \to (12)\sigma$. The only difference between σ and $\tau(\sigma)$ is that $\tau(\sigma(1)) = \sigma(2)$ and $\tau(\sigma(2)) = \sigma(1)$. If 1 and 2 are in the same cycle of σ, the map τ will divide this cycle to 2 new cycles. On the other hand, if 1 and 2 are in 2 different cycles, τ will combine them to a new one. Since the number of cycles always change with one and we have the same total number of elements, the number of even cycles is also changing with one. The map is apparently reversible.



56. [**2**] Consider the RHS. We will show that choosing a random permutation corresponds to choosing a unit(x or 1) from each parenthesis. Let's take arbitrary $\omega \in \mathcal{G}_n$. If we remove all of the elements of ω except 1, it will be alone in its own cycle of the permutation. Indeed, initially, we have a single x, which will represent that we've chosen the element 1 to be in its own cycle. Now, let's add the element 2 and let's make a choice from the second parenthesis. The element 2 can be either in a new cycle, different from those of 1, or it can be in the same cycle as the element 1, right after the element 1. Continuing that way, when we choose from the i-th parenthesis, the corresponding element i can be in its own cycle different from the currently existing cycles(this corresponds to choosing x) or it can be right after each of the previous $i-1$ numbers(this corresponds to choosing one of the $i-1$ 1's). This way, we can obtain the arbitrary pemutation ω, because at each step we can actually set the position of the element i with respect to the positions of the elements before i in ω. But in this process, the number of x's chosen corresponded to the number of cycles we have. Thus the coefficient in front of x^k will count the number of all permutations with exactly k cycles.

57 [**2**] We will prove that $f_{nk} = (n-1)!$ for any $k = 0, 1, \ldots, n$. If $\mathcal{G}'_{n-1}$ is the set of permutations of $[2..n]$ and $\mathcal{G}^k_n$ is the set of permutations of $[1..n]$ for which the length of the cycle containing 1 is k, then we have to come up with a bijective map $\tau : \mathcal{G}'_{n-1} \to \mathcal{G}^k_n$. Let ω be arbitrary permutation in $\mathcal{G}'_{n-1}$ and let $S = \{1, \omega(1), \omega(2), \ldots, \omega(k-1)\}$. If $\tau(\omega) = \sigma \in \mathcal{G}^k_n$, then define $\sigma(1) = \omega(1)$, $\sigma(\omega(1)) = \omega(2)$, $\sigma(\omega(2)) = \omega(3)$, $\ldots$,$\sigma(\omega(k-1)) = 1$. Then, if $[n] \setminus S = \{b_1 < b_2 < \ldots < b_{n-k}\}$, then define $\sigma(b_i) = \omega(n-i)$, for $i = 1, 2, \ldots, n-k$. In other words, we used the fact that considering the permutations for which the length of the cycle containing 1 is k, we have a natural ordering of the remaining $n-1$ elements - first those in the cycle of 1(ordered by the distance to the element 1) and then the rest taken in ascending or descending order(of course some other orderings and maps will work). At the end, we see that the probability $p_{nk} = \frac{1}{n}$, which does not depend on the value of k ($k = 0, 1, \ldots, n$).

58 (a) [**2**] The map $\tau : \sigma \to (12)\sigma$, used also in the solution of problem 55, is actually a bijection between the set of permutations of $[1..n]$ having 1 and 2 in the same cycle and

those having 1 and 2 in different cycles. See part (b) for generalization.
(b) [2+] Our goal is to proof that the number of permutations suffiicing the given conditions is $\frac{n!}{k!}\prod_{i=1}^{i=l}(\lambda_i-1)! = n(n-1)\dots(n-k+1)\prod_{i=1}^{i=l}(\lambda_i-1)!$. In how many different ways can we construct such a permutation σ? Let's first determine the order of the given k elements in each of the corresponding l cycles. This can be done in $\prod_{i=1}^{i=l}\frac{\lambda_i!}{\lambda_i} = \prod_{i=1}^{i=l}(\lambda_i-1)!$ ways. WLOG, we'll assume that this order is the natural one, i.e. $1,2,\dots,\lambda_1$, $\lambda_1+1,\lambda_1+2,\dots,\lambda_2$, and so on, up to $\lambda_{l-1}+1,\lambda_{l-1}+2,\dots,\lambda_l$. Then, let's choose predecessors(the elements $\sigma^{-1}(j)$) for each of the $n-k$ remaining numbers $j=k+1,k+2,\dots,n$. This can be done in $n(n-1)\dots(n-k+1)$ ways. It remains to determine $\sigma^{-1}(i)$ for $i=1,2,\dots,k$. Currently, we don't have values for them. Assign to $\sigma^{-1}(1)$ the last element having a value in the sequence λ_1, $\sigma(\lambda_1)$, $\sigma(\sigma(\lambda_1))$, $\sigma(\sigma(\sigma(\lambda_1)))$,... . Then assign to $\sigma^{-1}(2)$ the last element having a value in the sequence 1, $\sigma(1)$, $\sigma(\sigma(1))$, $\sigma(\sigma(\sigma(1)))$,... . Do this for $1,2,\dots,k$ and at the end σ will be uniquely determined. Using this algorithm, every such permutation is constructed exactly once.
(c) [3-]

59 [*] See section 4 of: Robert Cori, Michel Marcus, Gilles Schaeffer. Odd permutations are nicer than even ones. European Journal of Combinatorics, Elsevier, 2012, 33 (7), pp.1467-1478.

61 [2+] First note that a_1 is always a record. Take a permutation $\omega = a_1a_2\dots a_n$ with k records with indexes $1 = i_i < i_2 < \dots < i_k$. Let's construct a permutation σ with k cycles by the given ω. Let the first cycle of σ consists of the elements $a_1,a_2,\dots,a_{i_2-1}$ and $\sigma(a_1)=a_2$, $\sigma(a_2)=a_3$,..., $\sigma(a_{i_2-1})=a_1$. Let another cycle be $a_{i_2},a_{i_2+1},\dots,a_{i_3-1}$ and $\sigma(a_{i_2})=a_{i_2+1}$, $\sigma(a_{i_2+1})=i_2+2$,..., $\sigma(a_{i_3-1})=a_{i_2}$ and so forth. In other words, let the elements of ω between two paticular records belongs to the same cycle together with the left of the two records. Conversely, given a permutation σ with k cycles, we can simply sort these cycles in increasing order of their largest element and then list the elements in a new sequence(permutation) ω sequentially, beginning by the largest element in each cycle. Example: $\sigma = (461)(25)(3) \to 352614 = \omega$. This is Proposition 1.3.1 in EC2.

63 [2+] Let's set $\omega_n = \omega = a_1a_2\dots a_n$. Remove the largest letter, n, from ω_n to obtain ω_{n-1}. Continue this process until reaching $\omega_m = 12\dots m = id$. Set $\tau_m = \omega_m$. If ω_{m+1} is obtained from ω_m by inserting $(m+1)$ at the end of τ_m, then insert $(m+1)$ at the end of τ_m to obtain τ_{m+1}. Otherwise, if $(m+1)$ is obtained from τ_m by inserting $(m+1)$ before k in τ_m then put $(m+1)$ in the k-th place in τ_m and move what was in that place to the end of τ_m to obtain τ_{m+1}. Continue this process until reaching $\tau = \tau_n$ and define $\phi : \mathcal{G}_n \to \mathcal{G}_n$ by $\phi(\omega) = \tau$. We'll show that the number of excedances in τ_k equals the number of descents in ω_k for each $k = m, m+1, \dots, n$(see the explanations after the example). Also, it is obvious how to reverse the procedure to go from τ_n to ω_n. Just remove the largest letter and put the last one on its place until you reach $\tau_m = \omega_m$. Then, put back $m+1$, $m+2$,..., n consecutively before the numbers corresponding to the places of the same letters in $\tau_{m+1}, \tau_{m+2}\dots\tau_n$. We conclude that ϕ is a bijection with the needed properties.

Example: $\omega = 35142$.Here is how to reach τ:

$$\begin{array}{c} \mathbf{35142 \rightarrow 3142 \rightarrow 312 \rightarrow 12} \\ \hfill \downarrow \\ \mathbf{54123 \leftarrow 3412 \leftarrow 321 \leftarrow 12} \end{array}$$

For $\omega_m = \tau_m = id$ we have 0 descents and 0 excedances. Let j be between a and b in $\omega_j[..ajb..]$ for any $j > m$. We have to put j at position $b < j$ in τ_j. There are 2 cases. The first one is $desc(\omega_j) = desc(\omega_{j-1})$. Then $a > b$ and $a > m$, thus a is at position b in τ_{j-1}(a was placed there earlier in the process). This means that one excedance in τ_{j-1}(a¿b) is replaced with another one in τ_j ($j > b$). By moving a to the last position we ensure that we do not have a new excedance there. The second case is $desc(\omega_j) = desc(\omega_{j-1}) + 1$. This means that $a < b$, so before b in τ_{j-1} is either a or $b - 1$, i.e. the excedance $j > b$ in τ_j increases the number of excedances with 1.

Note: This is a proof described in the PhD thesis of Einar Steingrimsson from 1992 entitled "Permutations Statistics of Indexed and Poset Permutations".

64 [**2-**] Take a permutation $\omega = a_1a_2\ldots a_n$ with $k-1$ excedances and substract 1 from each $a_i > 1$ and instead of 1 write n. You get another permutation with k weak excedances, because all of the previous $k-1$ excedances are also weak excedances in the new permutation. In addition, those a_i with value 1 was not an excedance in the original permutation, while the n on its place in the new permutation must be a weak excedance. There are no other weak excedances in the newly obtained permutation, so their count is exactly k. The inverse map is obvious - add 1 to each element of a permutation with exactly k weak excedances(instead of n, write 1).

65 [**3-**] Note: The formulation given in the current version of the problem's list doesn't make sense. Some variables in the numerator of the RHS and in the hint should be exchanged with others. The right formulation was most probably the following:

Let $k \geq 0$. Then

$$\sum_{n\geq 0} n^k x^n = \frac{\sum_{i=1}^{k} A(k,i)x^i}{(1-x)^k}$$

Hint: Given a permutation $a_1a_2\ldots a_k \in \mathbb{G}_k$, consider all functions $f : [k] \to [n]$ satisfying $f(a_1) \leq f(a_2) \leq \ldots \leq f(a_k)$ and $f(a_i) \leq f(a_{i+1})$ if $a_i > a_{i+1}$

Solution: We can write the RHS in the form $(\sum_{i=1}^{k} A(k,i)x^i)(1+x+x^2+\ldots)^{k+1}$ in order to see more clearly that the coefficient in front of x^n on the right will be $\sum_{i=1}^{k} A(k,i)\binom{n-i+k}{k}$. Recall that $A(k,i)$ was the number of permutations in $\mathbb{G}_k$ with exactly $i-1$ descents. Also, $\binom{n-i+k}{k}$ is the number of different solutions of the equation (*) $b_0 + b_2 + \ldots + b_k = n - i$, where $b_i \geq 0$. We must prove then that $\sum_{i=1}^{k} A(k,i)\binom{n-i+k}{k} = n^k$. But, n^k is the number of different functions $f : [k] \to [n]$. Now, let's use the hint. Take a permutation $a_1a_2\ldots a_k$ with $i-1$ descents(positions j s.t. $a_j > a_{j+1}$) and a solution $b_0, b_1, \ldots, b_k$ of (*). Our goal is to determine $f(a_1), f(a_2), \ldots, f(a_k)$. We know that $f(a_1) + (f(a_2) - f(a_1)) + \ldots + (f(a_k) - f(a_{k-1})) + (n - f(a_k)) = n$, i.e. these are $k+1$ non-negative numbers with sum n. Yet, the numbers in the LHS of (*) sum up to $n - i$. However, we know that $f(a_1) \geq 1$ and also that $f(a_{j+1}) - f(a_j) \geq 1$ for all descents of the given permutation(we have the exact

descents). Since the number of descents is $i-1$, we get i of the considered numbers that have value at least 1. This shows us what to do. Add 1 to b_0 and to each b_j, where j is a descent($a_j > a_{j+1}$). Then take $f(a_1) = b_0$, $f(a_2) - f(a_1) = b_1$, $\ldots n - f(a_k) = b_k$. This determines completely a unique function $f : [k] \to [n]$.
On the other hand, if we are given a function $f : [k] \to [n]$, we can determine a unique permutation $\sigma = a_1 a_2 \ldots a_k \in \mathbb{G}_k$ and a solution of (*) for a particular i. Indeed, by f, we can determine the sets of indices $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(n)$. The first elements of σ must be the indices in $f^{-1}(1)$, then those in $f^{-1}(2)$ should follow and so forth. We know also that in σ, if you have a descent, then the function jumps. The last fact imply that the elements of each $f^{-1}(j)$ must be ordered ascendingly in σ, hence σ is determined completely. We can count the number of descents $i-1$ in σ, to assign $b_0 = f(a_1)$, $b_1 = f(a_2) - f(a_1)$, $\ldots b_k = n - f(a_k)$ and finally to substract 1 from b_0 and from each b_j, for which j is a descent in σ. We get a solution of (*) for the found i.

2. Permutations

67 [**3-**] Consider those S with exactly k elements and observe that the number of tuples (S, ω) with the given property will be $\binom{n}{k}\binom{n-k}{k}k!(n-k)! = n!\binom{n-k}{k}$. Indeed, choose the elements of S in $\binom{n}{k}$ ways, then their images under ω in $\binom{n-k}{k}k!$ ways(these elements must map to some of the other $n-k$ elements) and finally the images for all the remaining $n-k$ elements in $(n-k)!$ ways. Thus it suffices to prove that $\sum_{l\geq 0}\binom{n-k}{k} = F_{n+1}$. By problem 30, it suffices to prove that $\sum_{l\geq 0}\binom{n-k}{k}$ is the number of compositions of n with parts 1 and 2. But $\binom{n-k}{k}$ is just the number of such compositions with exactly k parts of size 2. Indeed, you have $n-k$ parts in total. If you place them one after another and choose the positions of those k of size 2, this choice maps to a composition of n of the given kind. Conversely, given an n-compositions with k 2s and $n-k$ 1s, we just take the numbers of those parts that are 2s and this determines uniquely our choice among all the possible $\binom{n-k}{k}$ such choices of parts.

68 [**1**] After taking one of the $n!$ permutations of all the n numbers, we can notice that after shufling the i_1th positions of the 1's in $i_1!$ ways, we get the same permuttion of the multiset. Using the same argument for the other numbers greater than 1, we see that any permutations of this multiset is counted $i_1! i_2! \ldots i_k!$ times.

69 [**2**] The multinomial coefficient $\binom{n}{b_1, b_2-b_1, b_3-b_2, \ldots, b_{k-1}-b_{k-2}, n-b_{k-1}}$ represent the number of ways to choose the first b_1 elements of ω and to order them in increasing order to get $\omega(1) < \omega(2) < \ldots < \omega(b_1)$. Then, to choose the next $b_2 - b_1$ elements of ω from the remaining pool of elements in $[1..n]$ and to order them in increasing order to get $\omega(b_1+1) < \omega(b_1+2) < \ldots < \omega(b_2)$ and so on. Obviously, we could have a descent $\omega(i) > \omega(i+1)$ only if $i \in S = \{b_1, b_2, \ldots, b_k\}$ since the other inequalities between the consecutive elements of ω are already determined. Conversely, it is clear that any particular permutation with descent set $S' \subseteq S$ is counted by this coefficient since we can just choose the first b_1, $b_2 - b_1$ elements, etc. exactly as they appear in the given permutation. Hence the map is onto and we are done.

70 [**3-**] This result was first proved by MacMahon in 1916. For the combinatorial proof, we need a special kind of bijection ϕ that is exchanging the inv and maj statistics, i.e. we are looking for $\phi : \mathbb{G}_n \to \mathbb{G}_n$, such that $inv(w) = maj(\phi(w))$ and $maj(w) = inv(\phi(w))$. Such bijection was first introduced by D.Foata in "On the Netto inversion number of a sequence", 1968. The same bijection is described in Section 1.4. of EC2.
Another one, $h(n)$, can be obtained from the composition of two bijections described in Sections 6.5 and 6.6 of the book "Bijective Combinatorics", N.Loehr, 2016. The first, $f_n : \mathbb{G}_n \to T_n$, where $T_n = [1] \times [2] \times \ldots [n]$, establishes that $\sum_{w\in\mathbb{G}_n} q^{inv(w)} = [n]!_q$. We already considered it in Problem 45: $f_n(w) = (t_1, t_2, \ldots, t_n)$, where $t_k = |\{(i,j) \in Inv(w) \mid w_i = k\}|$ are the number of inversions of w for which k is the bigger of the two involved elements. Now, for the second bijection, we can take the inverse of the map $g_n : \mathbb{G}_n \to T_n$ from the proof of Theorem 6.29 in Section 6.6 which states that $\sum_{w\in\mathbb{G}_n} q^{maj(w)} = [n]!_q$. This inverse map suggests to look at each t_k as $maj(w^{(k)}) - maj(w^{(k-1)})$, where $w^{(i)}$ would be the word obtained from w by erasing all letters larger than i. Thus, t_i records the extra contribution to maj caused by the insertion of the new symbol i. Look at the forementoned sections for description of the inverse map and proofs that those two are indeed bijections.

71 [**3**] This is dicussed in Section 1.4. of EC2, which follows the paper of D.Foata and M.Schutzenberger "Major Index and Inversion Number of Permutations", 1978.
First, they note that for any statistic f on $\mathbb{G}$, if we define $g(w) = f(w^{-1})$, then (f, g) has symmetric joint distribution(clearly $w \to w^{-1}$ is a map that proves this). They also show that if $imaj(w) = maj(w^{-1})$, then the bijection ϕ, with which they proved problem 70, preserves the $imaj$ statistic, i.e. $imaj(w) = imaj(\phi(w))$ (it even preserves the indexes of descent). Thus we have that $(maj, imaj)$ have symmetric joint distribution and ϕ transforms maj to inv while preserving $imaj$. This implies that $(inv, imaj)$ have symmetric joint distribution since the map $\phi \to \phi^{-1}$ exchanges maj and $imaj$, but preserves inv(see problem 46). Indeed, if you take w with, say, $inv(w) = 4$, $maj(w) = 5$ and $imaj(w) = 10$, then one can check that $\psi(w) = [\phi((\phi^{-1}(w^{-1}))^{-1})]^{-1}$ will be a permutation with $inv(\psi(w)) = 10$ and $imaj(\psi(w)) = 4$.

72 [**2**] First, note that in every alternating permutation, the position of 1 must be even and the position of n must be odd. Consider the alternating permutations $\omega = a_1a_2\ldots a_{n+1} \in \mathcal{G}_{n+1}$. If 1 is at position $k+1$ i.e. $a_{k+1} = 1$, then the elements in the set $S_k = \{a_1, a_2, \ldots a_k\}$ can be chosen in $\binom{n}{k}$ ways. The number of alternating permutations for S_k is the same as those of $\{1, 2, \ldots, k\}$, because the only thing that matters is the ordering of the numbers. Hence we have E_k ways to form the prefix before 1 and similarly E_{n-k} ways for the suffix. Thus we obtained that $E_{n+1} = \sum_{\substack{k=0 \\ k-even}}^{n} \binom{n}{k} E_k E_{n-k}$. In the same manner, by considering the position of $n+1$, we can obtain $E_{n+1} = \sum_{\substack{k=0 \\ k-odd}}^{n} \binom{n}{k} E_k E_{n-k}$ and now it remains to sum up the two equations to get what we want.

73 [**2+**] See the second proof of Theorem 1.1 in "A Survey of Alternating Permutations", R.Stanley, 2000.

74 [**2+**] Denote the LHS with $\sum_{k\geq 0} c_k x^k$ and the RHS with $\sum_{k\geq 0} d_k x^k$. We have $c_0 = E_0 \frac{x^0}{0!} = 1 = d_0$. Also, $c_k = d_k = 0$ for all odd k. Thus it suffuces to prove that $c_{2k} = d_{2k}$ for all $k \geq 1$. After substituting the derived expression for $tanx$ and $secx$ in problem 73 and simplifying, we obtain

$$c_{2k} = \sum_{l=0}^{k-1} \frac{E_{2l+1}}{(2l+1)!} \frac{E_{2k-2l-1}}{(2k-2l-1)!} = \sum_{l=0}^{k-1} \frac{1}{(2k)!} \binom{2k}{2l+1} E_{2l+1} E_{2k-(2l+1)}$$

Similarly,

$$d_{2k} = \sum_{l=0}^{k} \frac{E_{2l}}{(2l)!} \frac{E_{2k-2l}}{(2k-2l)!} = \sum_{l=0}^{k} \frac{1}{(2k)!} \binom{2k}{2l} E_{2l} E_{2k-2l}$$

But using the proof of problem 72, we can see that both expressions written above are equal to $\frac{1}{(2k)!} E_{2k+1}$ which finishes the proof.

75 [**2+**] See exercise 5.7.b in [6]

76 [**2**] The problem's formulation assumes that $k|n$. Let's descibe a process that constructs all such permutations. Start from the element 1. It belongs to a cycle with length divisible by k. We can determine its successor $\omega(1)$ in $n-1$ ways, $n-2$ ways for $\omega(\omega(1))$ and so on. Assume, that we have already determined $k-1$ elements of this cycle. For the k-th element, we have all the remaining $n-k$ elements plus the element 1, since we are allowed to close the cycle at this position. This makes $n-k+1$ ways for the k-th element instead of $n-k$ ways. If we close the cycle here, we can continue from an arbitrary element among the remaining, not currently used elements. Pick one and observe that its successor could be determined in (n-k-1) ways, so the multiplier (n-k) is missing. Otherwise, if we do not close the cycle after putting the first k elements into it, then the ways of assigning the next successor are again $n-k-1$ since we have already used k elements and we are not able to close the cycle at this next moment. We have the same situation at each moment divisible by k, when we could either close the cycle or continue without closing it. We get the answer by multiplying the possible number of ways at each step.

77 [**3**] Look at the proof of Theorem 2 in the note "Counting Permutations", E.Bolker, A.Gleason, 1980.

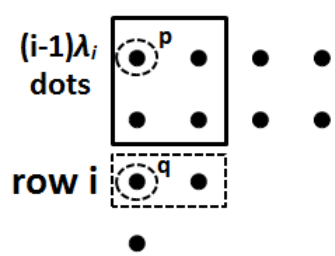
78 [**2**] The condition $f^j = f^{j+1}$ for some $j \geq 1$ implies that $f(f^j) = f^j$, i.e. the image of f^j for some j, consists entirely of fixed points of f. Given f, construct a labeled tree T with $n+1$ vertices in the following way: Denote the set of fixed points for f with F. Start from a vertex with label $n+1$ and take $L_1 = F$ to be the set of the neighbours of $n+1$. Then take $L_2 = f^{-1}(L_1)$ to be the second level in the tree T, with corresponding edges $E_1 = \{(i,j) \mid i \in L_1, f(j) = i\}$. Continue this way by taking $L_k = f^{-1}(L_{k-1})$ and by adding the edges $E_k = \{(i,j) \mid i \in L_{k-1}, f(j) = i\}$, until all the vertices with labels in $[n]$ occur. This must happen at some point since $f^j = f^{j+1}$ for some j, as we explained. The fact that T is a tree could be checked easily. Also, given a tree T, we can easily determine f by looking at its edges and by interpreting an edge $(i, n+1)$ as "f(i)=i", for every such i. Verify that the correspondence is 1-to-1.

79 **[2]** The image of $f(x)$, i.e. $I = \{f(x) \mid x \in [n]\}$ must be consisted solely of fixed points of f, since $f(f(x)) = f(x)$. Let i determines the number of the fixed points for f. We can select them in $\binom{n}{i}$ ways. In order to obtain f, it suffices to determine the images for the remaining $n-i$ non-fixed points. This can be done in i^{n-i} ways, because every such element must go to a fixed point. Sum up for i from 1 to n to get the answer(permutations without fixed points cannot satisfy $f = f^2$).

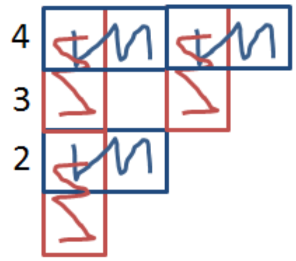
80 **[2+]** The condition $f = f^j$ for some $j \geq 2$ implies that after certain number of compositions, f applied on the set $I = \{f(x) \mid x \in [n]\}$, becomes identity. The later would be true if and only if each element of I goes back to itself after applying f certain number of times. First note, that $f(I) \subseteq I$. If $f(I) = I$ i.e. if f restricted on I is a permutation(bijection), then each element of I belongs to a cycle in this permutation, so f would satisfy the condition since $j = \prod_{i \in I} r_i$ would works, where the cycles of the elements of I have cardinalities r_i. Alternatively, one can just use that the order of each permutation is finite. Otherwise, if f restricted on I is not a bijection, then we would have $i, j \in I$ for which $f(i) = f(j) = k$ and when we consider the sequence $k, f(k), f(f(k)), \ldots$ either i or j occurs first, or neither. In all these cases we will have at least one element y of I for which $f^j(y) \neq y$, for all $j \geq 2$(i.e. y does not occur in the sequence). Indeed, if for instance i occurs first, then j does not occur in the sequence at all(try to see why). Therefore, we're looking for the number of functions f, which are permutations when restricted to its image set I. Now, we can see that each term in the given expression for $h(n)$ can be obtained as follows: Choose the set I with $|I| = i$ in $\binom{n}{i}$ ways, then choose a permutation of those i numbers in $i!$ ways. Finally, choose the image for the remaining $n-i$ elements $x \notin I$ in i^{n-i} ways. Sum up by i from 1 to n.

3. Partitions

87 [1+] Consider the corresponding Ferrer's diagram for the partition. The RHS gives the number of ways to choose 2 dots, p and q, in the same column of this diagram. WLOG, let q be below p on the diagram. The LHS represents a partitioning of the same sets of choices by the row of q(the part of λ that it belongs to). Indeed, if q belongs to the part λ_i, then we have exactly $(i-1)\lambda_i$ posibilities for p and once we know p and the number of the part - i, q is uniquely determined.



88 [1+] Consider the first equality and the figure below, where the partition $\lambda = (4, 3, 2)$ was taken. Take all the squares in odd rows and odd columns and draw a 2-block containing each of them to the right(in blue) and another 2-block to the bottom(in red). Note that the LHS counts the number of blue blocks and the RHS counts the number of red blocks. The second and the third equality have analogous proofs by considering 2-blocks for each square in odd rows, even columns and even rows, odd columns, respectively.



89 [1] If a partition has largest part k, then the conjugate partition has exactly k parts. Conversely, if a partition has exactly k parts, then the conjugate partition has largest part k(note that maping a partition to its conjugate is an involution).

90 [2+] Let S be the set of partitions of n. Given some $\lambda \in S$, one can obtain another partition by taking $i\ k's$ from λ and replacing them by $k\ i's$.

$$\underbrace{k\,k\ldots k}_{i} \to \underbrace{i\,i\ldots i}_{k}$$

Consider a fixed k and note the following:

- The number of partitions that can be obtained from λ is $f_k(\lambda)$. This is because one can choose any $i \in \{1, 2, \cdots f_k(\lambda)\}$, an then transform $i\ k's$ into $k\ i's$.
- The number of partitions that can lead to λ is $g_k(\lambda)$. This is because if a value i in λ is repeated at least k times, λ can be obtained from the partition that replaces k of those $i's$ with $i\ k's$.

Knowing this, imagine a directed graph G where the nodes represent a partition of n and there is a directed edge (λ, λ') if one can obtain λ' from λ. In this graph, it is clear that $d_{out}(\lambda) = f_k(\lambda)$ and $d_{in}(\lambda) = g_k(\lambda)$. Therefore,

$$\sum_{\lambda \vdash n} f_k(\lambda) = \sum_{\lambda \in G} d_{out}(\lambda) = \sum_{\lambda \in G} d_{in}(\lambda) = \sum_{\lambda \vdash n} g_k(\lambda)$$

91 [2+] Let σ be a singleton in a partition λ of $n-1$, and consider the Young diagram of σ. Replacing σ by $\sigma+1$, deleting columns σ and $\sigma+1$, and conjugating the resulting diagram gives a partition of $n-2k$ for some $1 \le k \le \frac{n}{2}$. Conversely, given a partition λ of $n-2k$ for some $1 \le k \le \frac{n}{2}$, adding two consecutive rows of k squares, deleting the bottommost square in the k-th column and conjugating the resulting diagram yields a singleton in row k. Note that both processes are injective and inverses of each other, hence the map is a bijection.

For each partition of λ of n, enumerate the 2's that appear in λ as $2_1, 2_2, \ldots, 2_k$, where $k(\lambda)$ is the multiplicity of 2 in λ. Associate 2_j in the partition λ with the partition of $n-2j$ obtained by deleting j rows of 2's. For a partition λ of $n-2j$ for some $1 \le j \le \frac{n}{2}$, associate λ with the element 2_j of the partition $\tilde{\lambda}$ obtained by adding j elements of 2 to λ. Again, these processes are injective and inverses of each other, so the map is a bijection. Composing this map with the previous map yields the desired bijection.

A nice generating function proof can be found here, which loosely inspired the bijective proof presented here.

92 [2-] If $\lambda_1 = \lambda_2$, combine these two parts, otherwise divide λ_1 into 2 equal parts. More formally, consider the set $P^{(2)}$ of the partitions of n, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ into powers of 2. WLOG, we can assume that $\lambda_{k+1} = \lambda_{k+2} = \ldots = 0$. Represent this set $P^{(2)}$ as a disjoint union of the sets Q_1 and Q_2, where Q_1 is the set of those partition in $P^{(2)}$ for which $\lambda_1 = \lambda_2$ and Q_2 is the set of those partitions in $P^{(2)}$ for which $\lambda_1 \neq \lambda_2$. It is easy to see that $\mid Q_1 \mid = \mid Q_2 \mid$ since the map

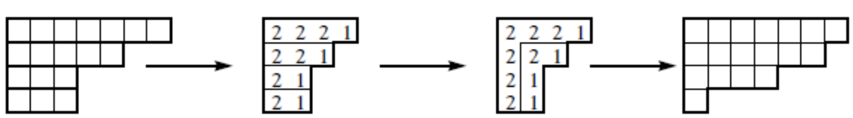
$$\phi((\lambda_1, \lambda_1, \ldots, \lambda_k)) = (2\lambda_1, \ldots, \lambda_k)$$

is a bijection between Q_1 and Q_2. Indeed, it is obviously injective and also onto since if we have some $\lambda = (2\lambda_1, \lambda_2, \ldots, \lambda_k) \in Q_2$ for some $n \geq 2$, then $(\lambda_1, \lambda_1, \lambda_2, \ldots, \lambda_k)$ is apparently a partition since from $2\lambda_1 > \lambda_2$ and the fact that $2\lambda_1$, λ_2 are powers of 2, follows that $\lambda_1 \geq \lambda_2$.

93 [1] Consider the map $\phi((\lambda_1, \lambda_2, \ldots, \lambda_k)) = (\lambda_1 + 1, \lambda_2 + 2, \ldots, \lambda_k + k)$ from the set of all partitions of n with k parts to the set of partitions of $n + \binom{k}{2}$ with k parts. The resulting partition always has distinct parts, because even if some of the parts of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ are equal to each other, the map makes the difference between the corresponding parts in the new partition to be 1. The inverse map is simply $\phi^{-1}((\lambda_1, \lambda_2, \ldots, \lambda_k)) = (\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_k - k)$.

94 [2] Take arbitrary partition of n into odd parts and write it in the form $n = a_1 \cdot 1 + a_3 \cdot 3 + \ldots$. Then, write each a_j as a sum of powers of two, using its binary representation to get $n = (2^{b_{11}} + 2^{b_{12}} + \ldots) \cdot 1 + (2^{b_{31}} + 2^{b_{32}} + \ldots) \cdot 3 + \ldots$. What remains is just to foil out the paranthesis to obtain sumation of terms of the kind $2^x \cdot y$, for odd y. Since each number has unique such representation, we will have different parts. One simple example is $14 = 5 + 3 + 3 + 1 + 1 + 1 = (1)5 + (2)3 + (3)1 = (1)5 + (2)3 + (2 + 1)1 = 5 + 6 + 2 + 1$. Conversely, given a partition into distinct parts, we can write each part as a power of 2 multiplied by an odd number, and collect the coefficients of each odd number, and write the odd number those many times, to get a partition into odd parts. This bijection and the nice generating functions proof of Euler is described in this math.stackexchange topic.

In addition, three different bijections without proofs, different from the one above, are shown in the work of Igor Pak "Partition Bijections, a survey". Here, we give a proof of the third bijection mentioned there. Let us have a partition λ consisted only of odd parts. Draw a 2-modular diagram corresponding to λ(look at the figure below)



Draw successive hooks $H_1, H_2, \ldots$, as shown in the figure. Let μ_1 be the number of squares in H_1, let μ_2 be the number of 2s in H_1, let μ_3 be the number of squares in H_2, let μ_4 be the number of 2s in H_2, etc. Each hook contains the end of a certain part of the partition and since each part is odd, it ends up with a 1, thus every hook has at least one 1 and the number of 2s in a hook is less than the number of squares. Also, one can easily see that the number of 2s in hook H_i is greater than the number of squares in H_{i+1}. The latter is true since for each square s in H_{i+1}, we have a 2 from H_i next to it (either on the left or above). You cannot have a square s such that the part to the left of s or above s is 1, but not 2. With this we just showed that the resulting partition μ is indeed a partition and has distinct parts. The reverse map can be constructed easily. One should just see that each hook is uniquely determined by the number of 2s(or respectively 1s) in it, because we know that the row which is part of the hook ends with a 1 and also that if there are other 1s, they must be at the end of the column part of the hook.

95 [2] The fact can be stated in more general form. The number of partitions of n with no part repeated more than d times equals the number of those partitions of n with no part divisible by $d+1$. This is the Glaisher's theorem and it generalizes the statement of problem 94. Below is a bijective solution posted in *math.stackexchange* as part of the discussion here :
On the side without too many repetitions, break up parts divisible by d^k into d^k parts (where d^k is the highest power of d dividing the size of the part). You receive a partition without parts divisible by d. For the other direction, write the multiplicities in base d and if you have $\sum_i a_i.d^i$ parts of size s, then you glue them to get a_i parts of size $s.d^i$.

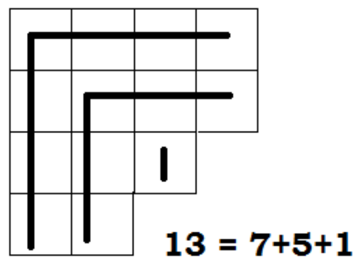
96 [2] Let A be the set of pairs described in the problem. Let B be the set of pairs (i, λ_i) where $i \in \{1, 2, \cdots, n+1\}$ and $\lambda_i \vdash (n+1-i)$. The number of elements in B is $p(0) + p(1) + \cdots + p(n)$ because for each i there are $p(n+1-i)$ partitions that can go with it.
We are going to show a bijection between A and B.
Take a pair $(\lambda, \mu) \in A$ and let's consider the Young diagram of μ. Let's consider the row that contains the extra square. If this row has i squares, we assign λ_i to the partition that results from removing this row, which has a total of $n+1-i$ squares. This way obtain a pair $(i, \lambda_i) \in B$.
The removed row must have been the last one with i squares in μ because otherwise, when removing the additional square, λ wouldn't be a valid partition. Therefore, one can transform the pair in B back to its corresponding pair in A by adding a row with i squares to λ_i right after the smallest row with at least i squares. This gives us μ, and removing the last square from this row gives us λ.

97 [2] It is clear that $np(n)$ is equal to the number of squares that appear among all Young Diagrams' for partitions of n. Consider a square s in a Young Diagram for some partition λ of n. This square will appear in the j-th column of the k-th row of the Young Diagram, where $1 \leq j \leq \lambda_k$. Let m be the number of rows of λ located beneath or at the same height as s equal in size to λ_k, and let $i = m\lambda_k$. To this square we associate a tuple $(j, \lambda_k, i, \lambda')$, where *lambda*$'$ is the partition of $n-i$ corresponding to deleting m rows of λ equal in size to λ_k. The mapping is injective and generates a tuple such that $1 \leq j \leq \lambda_k$, $\lambda_k | i$, and λ' is a partition of $n-i$.

Similarly, given a tuple (j, t, i, λ'), where $1 \leq j \leq t$, $t|i$, and λ' is a partition of $n-i$, we can insert $\frac{i}{t}$ parts equal to t into the partition λ' to obtain a partition λ of n. In this partition we associate the tuple to the square in the j-th column of topmost row added to

λ' (we assume new rows are added in the lowest position possible). This demonstrates that the mapping is indeed a bijection.

98 [2] The Young diagram of a self-conjugate partition can be decomposed into hooks with a diagonal square as corner(see the figure below), and can be reconstructed from the sequence of the (distinct odd) sizes of the hooks. As a consequence k(n) equals the number of partitions of n into distinct odd parts.



99 [2+] From p.98. we obtain that $k(n)$ equals the number of partitions into distinct odd parts denoted by $p_{DO}(n)$. To equate this number to $e(n) - o(n)$, consider the following involution(bijection which coincides with its inverse) that match a partition in the set $S = p(n) \setminus p_{DO}(n)$ to another element of S with the opposite parity of its number of its even parts. For $\lambda \in S$, test the odd numbers $m = 1, 3, 5, \dots$ in order, considering the set of parts of λ of the form $2^k m$ with $k \in \mathcal{N}$. If the set is either empty or consists of a single part m, move on to the next odd number. Since $\lambda \in S$, there is some m that is not skipped; stop at the smallest such m. Now take the maximal k such that λ has a part of size $2^k m$. If this part is unique, then $k \neq 0$ by the choice of m; break the part into two parts of size $2^{k-1}m$. The number of even parts either increases by 1(if $k > 1$) or decreases by 1(if $k = 1$). Note that even if $k - 1 = 0$, then the new partition is in S since it has 2 equal parts. On the other hand, if λ has at least two such parts: glue them together to form a part of size $2^{k+1}m$. The number of even parts decreases by 1. One readily checks that the partition λ so produced is in S and that this procedure is an involution on S. We already saw that λ, μ always have opposite parities for the number of their even parts.
This is the solution of Marc van Leeuwen from this math.stackexchange topic, together with a few additional remarks that were added.

100 [2] Look at p.218-221 from P.MacMahon, "Combinatory Analysis".

102 [3-] For any triangle with integer side lengths and perimeter n, we can associate to it a partition (a, b, c) of n such that $1 \leq a \leq b \leq c$ and $a + b > c$ (triangle inequality). Recalling that two triangles are congruent if and only if we can map the sides of one triangle to corresponding sides of the second triangle with equal length, it follows that all partitions (a, b, c) with $1 \leq a \leq b \leq c$ encode a unique triangle.

Consider the basis $B = \{(0, 1, 1), (1, 1, 1), (1, 1, 2)\} \subseteq \mathbb{R}^3$. If $\lambda = (a, b, c)$ is a partition of n corresponding to a triangle, then we can express the corresponding point $(a, b, c) \in \mathbb{R}^3$ as a linear combination of the basis elements in B with non-negative integer coefficients:

$$(a, b, c) = (b - a)(0, 1, 1) + (a + b - c)(1, 1, 1) + (c - b)(1, 1, 2)$$

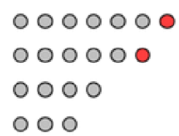
Since B is a basis, the coefficients are distinct for different triangles. To these coefficients we associate a partition of $n-3$ consisting of $b-a$ 2's, $a+b-c-1$ 3's, and $c-b$ 4's.

Given a partition of $n-3$ consisting of a 2's, b 3's, and c 4's, we can generate the partition $\lambda = (b+c+1, a+b+c+1, a+b+2c+1)$ of n. This partition satisfies the triangle inequality (hence encodes a triangle with integer side lengths of perimeter n) and is distinct based upon the choice of (a, b, c). Thus, this procedure describes a bijection between the set of incongruent triangles with integer side lengths of perimeter n, and partitions of $n-3$ consisting of only 2's, 3's, and 4's.

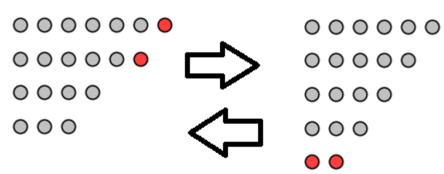
103. [**3**] First note that the equality can be rewritten as:

$$(1-x)(1-x^2)(1-x^3)\ldots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} \ldots$$

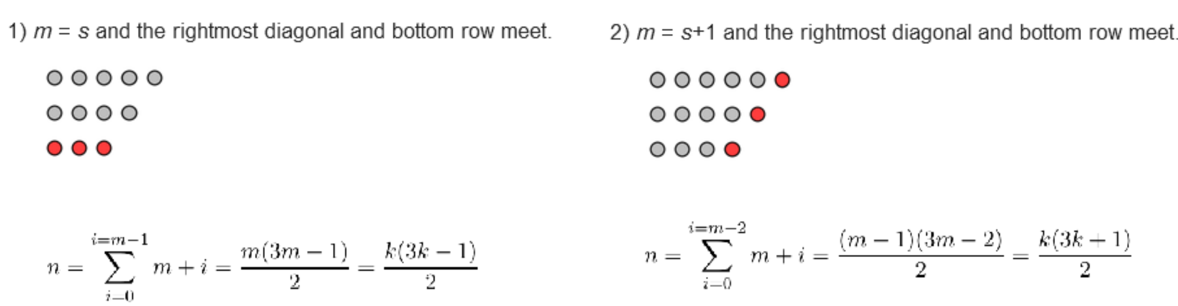
The exponents $1, 2, 5, 7, 12, \ldots$ on the right hand side are given by the formula $g_k = \frac{k(3k-1)}{2}$ for $k = 1, -1, 2, -2, \ldots$ and are called pentagonal numbers. Consider the Ferrer's diagram for a partition of n with distinct parts(see the figure below). Let m be the number of elements in the smallest row of the diagram ($m = 3$ in the example). Let s be the number of elements in the rightmost 45 degree line of the diagram ($s = 2$ in the example).



In general, if $s < m$(except some special cases described later), then we can take the s dots in the rightmost 45-degree line and move it at the bottom of the diagram to form a new row. Notice that the obtained diagram has the property $s \geq m$. This map also changes the parity of the number of parts for the partition and it's reversible, since when $s \geq m$, we can move the last row to form a rightmost diagonal(look at the figure below).



The only problematic cases when we cannot use the showed correspondence($\frac{k(3k\pm1)}{2}$) are given on the last figure and this explains the other part of the given formula:



For more details, see this Wikipedia article, which the current solution follows.

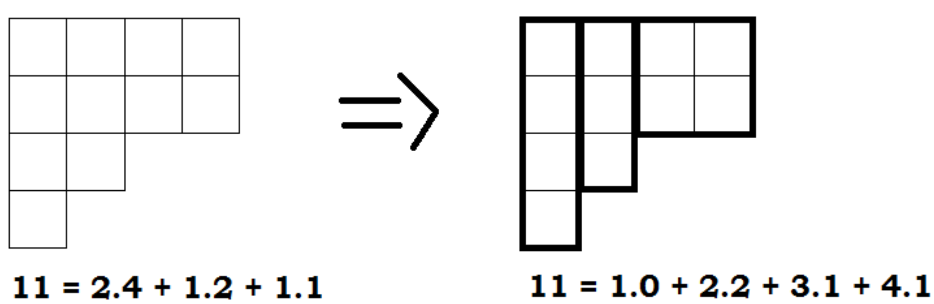
104 [2] Using the same bijection as the previous problem (see this Wikipedia article), note that with the exception of the two problematic cases, the involution maps a partition of n with an even largest part to a partition of n with an odd largest part, and vice versa. Thus, if n is not of the form $\frac{k(3k+1)}{2}$ or $\frac{k(3k-1)}{2}$ for some $k \geq 1$, then we have $f(n) - g(n) = 0$. If $n = \frac{k(3k+1)}{2}$ for some $k \geq 1$, then the only problematic partition has a Ferrer's diagram where the bottom row (length m) and rightmost diagonal (length s) meet, and $m = s + 1$. We see that the largest element has size $m + (s+1) - 1 = 2m$, so $f(n) - g(n) = 1$. If $n = \frac{k(3k-1)}{2}$ for some $k \geq 1$, then the only problematic partition has a Ferrer's diagram where the bottom row (length m) and rightmost diagonal (length s) meet, and $m = s$. We see that the largest element has size $m + s - 1 = 2m - 1$, so $f(n) - g(n) = -1$. The case for $n = 0$ is trivial, since the empty partition is considered to have an even largest part vacuously, so $f(0) - g(0) = 1$.

115 [2-] The coefficients in front of q^k on the left and on the right are both polynomials of x. *Equality 1.*
In order to follow the arguments below, let us write the equality in the form:

$$\prod_{i \geq 1} (1 + qx^i + q^2 x^{2i} + \ldots) = (x + x^2 + x^3 + \ldots)(x + x^3 + x^5 + \ldots) \ldots (x + x^{k+1} + x^{2k+1} + \ldots) \cdot q^k$$

For any $u \geq 0$, *the coefficient in front of* x^u *in this polynomial for the LHS will be given by the number of solutions* $g_1(u,k)$ *to* $a_1 i_1 + a_2 i_2 + \ldots + a_l i_l = u$ *over all compositions* $a_1 + \ldots + a_l = k$ *and where* $1 < i_l < i_{l-1} < \ldots < i_1$. *Note that this is exactly the number of partitions of* u *with* k *parts. We can simply substract 1 from each* i_j *to get that* $g_1(u,k)$ *is the number of solutions to* $a_1 i'_1 + a_2 i'_2 + \ldots + a_l i'_l = u - k = v$ *over all compositions* $a_1 + \ldots + a_l = k$ *and where* $0 < i'_l < i'_{l-1} < \ldots < i'_1$. *On the other hand, the coefficient in front of* x^u *for the RHS will be given by the number of solutions* $g_2(u,k)$ *to* $u = (l_1 + 1) + (2l_2 + 1) + \ldots + (kl_k + 1)$, *where* $l_j \geq 0$. *We can transform this to* $v = u - k = l_1 + 2 \cdot l_2 + \ldots + kl_k$. *Therefore in order to prove that* $g_1(u,k) = g_2(u,k)$, *it suffices to show the following*

Main fact. *For any* $v \geq 0$, *the number of partitions of* v *with at most* k *parts equals the number of solutions to* $v = l_1 + 2 \cdot l_2 + \ldots + kl_k$, *where* $l_j \geq 0$.
Proof: Observe that this follows easily from the decomposition of the Young's diagram for a partition to rectangles with maximum possible height, as shown in the example below:



In fact, this is the trivial correspondence between a partition and its conjugate. Formally, given a partition $v = a_1\lambda_1 + a_2\lambda_2 + \ldots + a_l\lambda_l$, *we can represent it as* $v = (\lambda_1 - \lambda_2) \cdot a_1 + (\lambda_2 - \lambda_3) \cdot (a_1 + a_2) + \ldots + \lambda_l(a_1 + a_2 + \ldots + a_l)$. *The inverse map is trivial.*

Equality 3.
Let's write the equality in the form:

$$\prod_{i\geq 1}(1+qx^i) = (x+x^2+x^3+\ldots)(x^2+x^4+\ldots)\ldots(x^k+x^{2k}+\ldots)\cdot q^k$$

For any $u \geq 0$, the coefficient in front of x^u in this polynomial for the LHS will be given by the number of solutions to $u = i_1 + i_2 + \ldots + i_k$, where i_j are distinct and positive. Now, we can use p.93 to see that this number is the same as the number of partitions of $u - \binom{k+1}{2}$ with at most k parts(simply substract j from each i_j). As for the RHS, the coefficient in front of x^u in it will be the number of solutions to $u = 1l_1 + 2l_2 + \ldots + kl_k$, where $l_j \geq 1$. Substract j from each l_j and summon the main fact.

Equality 4.
By a similar approach, we obtain that it suffices to prove that the number of solutions to $u = 2b_1 + 2b_2 + \ldots + 2b_k$, *where* $b_1 \geq b_2 \geq \ldots \geq b_k \geq 0$, *equals the number of solutions to* $u = 2c_1 + 4c_2 + \ldots + 2kc_k$, *thus we can again rely on the main fact.*

116 *[3] The identities in the previous p.115 could help one to prove the Jacobi triple identity, as described here. This is a generalization of the Euler Pentagonal Theorem. The first combinatorial proof was given by Sylvester and Franklin in 1882("A constructive theory of partitions, arranged in three acts, an interact and an exodion."). For a most recent proof(from 2016), see "A combinatorial proof of Jacobi's triple product identity' by L.Kolitsch and S.Kolitsch*

121 *[1] Every partition counted in* $p(j,k,n)$ *is contained in a rectangle with* j *rows and* k *columns. Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ *be a partition of* n *with* $l \leq j$ *and* $\lambda_1 \leq k$. *Obviously, the rest of the* $j \times k$ *rectangle which is uncovered by* λ *forms a partition of* $j \cdot k - n$. *We can get a formal definition of this partition by first adding some 0s at the end of* λ *in order to make its length to be* j, *i.e. assume that* $\lambda_{l+1} = \ldots = \lambda_j = 0$. *The new partition* $\mu = (\mu_1, \mu_2, \ldots, \mu_j)$ *can be obtained with the formula* $\mu_u = k - \lambda_{j+1-u}$, *for* $u = 1, 2, \ldots, j$. *Conversely, having a partition* μ *of* $j \cdot k - n$, *we can obtain partition* λ *of* n *by exactly the same approach - by looking at the parts that must be added to the parts of the given partition to get* k. *Formally stated,* $\lambda_u = k - \mu_{j+1-u}$, *for* $u = 1, 2, \ldots, j$, *i.e. the suggested map is an involution.*

124 *[1] Just get the partition of the missing numbers out of those in* $[k]$. *Formally, let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ *be a partition of* n *into distinct parts, with lagest part at most* k. *Take the elements of* $[k] \setminus \{\lambda_1, \lambda_2, \ldots \lambda_l\}$ *and list them in descending order to get a partition into distinct parts for* $\binom{k+1}{2} - n$. *The described map is an involution.*

126 *[2+] This proof follows the solution of exercise 108 in* [5].
Let P_{n-1} *be the set of partitions of [n-1].*
Let P'_n *be the set of partitions of [n] for which no block contains two consecutive integers.*
Let $p \in P_{n-1}$. *We are going to transform* p *into a partition* $p' \in P'_n$.
To construct p', *first we are going to add a block containing* n.
Let $i, i+1, \cdots, j$ *for* $i < j$ *be a maximal sequence of consecutive integers that is contained in a block of* p. *Then, we are going to move* $j-1, j-3, \cdots$ *from this block to the block containing* n. *We do this for all such maximal sequences. It is easy to see that the resulting*

partition belongs to P'_n.
Now, given p$'$ we can construct p as follows: let X be the block containing n. We discard X, and for each $i \neq n$ in X, we are going to add i to the block containing $i+1$.
As an example, the partition 1456-2378 in P_8 becomes 146-38-2579 in P'_9.

128 [3-] We'll count the number of rooted trees. Note that one can think of the edges of a rooted tree as directed edges that point towards the node's children. We are going to refer to the ends of a directed edge as head and tail as in the image:

One can construct a tree by starting out with the set of n vertices and sucessively adding the $n-1$ directed edges that belong to that tree. We note the following about the construction:

- At every step of the process the connected components of the graph must be rooted trees themselves, because otherwise the final graph won't be a rooted tree.
- When an edge is added, it joins 2 different connected components, decreasing the number of connected components by 1.
- When an edge is added, its head must be a node that still doesn't have a parent because in a rooted tree a node cannot have more than one parent.

Because of these three properties we know that before adding the i-th edge, the graph has $n-(i-1)$ rooted subtrees. One can choose any of the n nodes to be the tail of the edge. Once chosen, the head of the edge cannot be part of the same connected component, and must be a node without a parent. Therefore, it can only be one of the roots of the $n-i$ remaining subtrees. From this, one knows that there are $n(n-i)$ ways of choosing the $i-th$ edge, regardless of which edges where previously chosen. Therefore, there are $(n-1)!n^{n-1}$ sequences of directed edges that construct a rooted tree. Since any of the $(n-1)!$ permutations of a valid set of directed edges, represents the same rooted tree, the total number of rooted trees is n^{n-1}.

This result is known as Cayley's Formula. The idea behind the proof shown above was first thought by Pitman. Pitman's solution as well as other combinatorial proofs proposed by Prüfer and by Joyal can be found in R. P. Stanley, "Topics in algebraic combinatorics." Course notes for Mathematics 192 (2012): 168-171.
Finally, another very simple bijection used to enumerate rooted trees by creating a functional digraph is due to Ö. Eğecioğlu and J. Remmel, "Bijections for Cayley trees, spanning trees, and their q-analogues," J. Combin. Theory Ser. A 42 (1986), 15-30.

129 [1+] Let's consider the following bijection between planted forests on [n] and trees on [n+1] whose root is node $n+1$:
Given a planted forest on [n] one can construct a rooted tree on [n+1] by adding a node labeled $n+1$ and adding directed edges from this node to each of the roots of the trees that make up the planted forest. To go back, one simply removes the node labeled $n+1$, as well as all its incident edges.

Since the number of trees on [n+1] whose root is node $n+1$ is $(n+1)^{n-1}$ so is the number of planted forests on $[n]$.

130 *[2] To prove this, we'll double count T_k the number of planted forests with k subtrees. On one hand, one can choose the k nodes that are going to be the root set S in $\binom{n}{k}$ ways. Once we select the root set S, one can construct a planted forest in $p_S(n)$ ways. This gives us that:*

$$T_k = \binom{n}{k} p_S(n).$$

On the other hand, one can take an approach similar to the one in problem 128 and construct a planted forest with k components by successively selecting $n-k$ edges. This will give us $\prod_{i=1}^{n-k} n(n-i)$ sequences in which the $n-k$ directed edges can be chosen to construct the forest. Therefore, one must divide this value by the number of permutations in which a specific set of edges can appear, which is $(n-k)!$. From this, one gets that:

$$T_k = \frac{(n-1)(n-2)\cdots(n-(n-k))}{(n-k)!} n^{n-k} = \binom{n-1}{k-1} n^{n-k}.$$

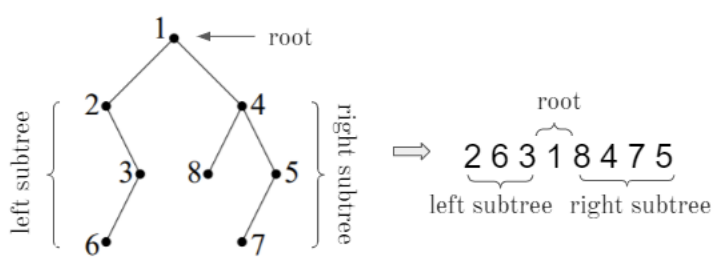
Setting equal the two values, and solving for $p_S(n)$, we get the desired result:

$$p_S(n) = kn^{n-k-1}.$$

131 *[2] See theorem 5.3.4 in* [6].

133 *b. [2]*

(1) *We'll show a bijection between the set of increasing binary trees on [n] and the set of permutations of [n].*
First, note that in an increasing binary tree, the smallest element has to be the root since it cannot be the child of any other node. Now, consider a permutation of [n]. In this permutation, all the elements to the left of the smallest value are going to make up the left subtree, and all the elements to the right are going to make up the right subtree. This way, one can recursively create an increasing binary tree by successively selecting the smallest values on each side to be the roots of the subtrees.
One can transform an increasing binary tree back to its corresponding permutation by doing a similar construction: one recursively transforms the left and right subtrees into ordered sequences and then takes the ordered sequence from the left subtree, followed by the root, followed by the ordered sequence of the right subtree.

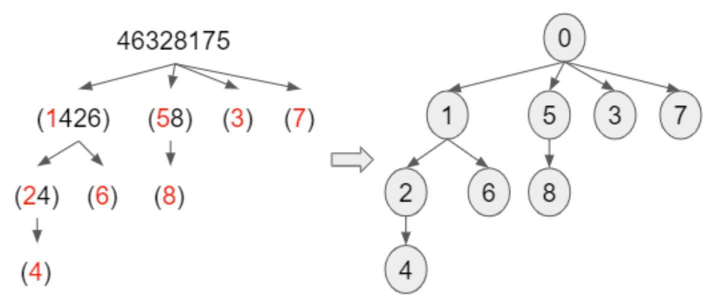


(2) *If there is a descent in a permutation, let's say between the values a and b, then it means that $a > b$ so a must belong to one of b's subtrees. If fact, a cannot be b's parent because it is greater than b, and it cannot belong to a branch different from*

b's because they wouldn't be neighbors (there would be at least one predecessor node separating them). Since a is to the left of b it must belong to the left subtree, hence the existence of a left child of b. Therefore, ***the number of left child is at least as big as the number of descents.***
If node b has a left child, then all the nodes belonging to this left subtree must be greater than b. Therefore in the corresponding permutation, there will be a sequence of nodes to the left of b all of which are greater than b, so there is at least a descent (between b and its left neighbor). Therefore ***the number of descents is at least as big as the number of left childs.***
Therefore the number of left children in an increasing binary tree must be equal to the number of descents in its corresponding permutation, which implies that the number of increasing binary trees with k left children is $A(n, k+1)$.

134

(1) *We define an increasing tree as a tree where every node is smaller than its children. There is an obvious bijection between the set of increasing forests on [n] and the set of increasing trees on [n+1] since we can transform an increasing forest of n vertices into an increasing tree of $n+1$ vertices by adding a 0-th node that has directed edges towards each of the nodes in the forest's root set.*
Now we are going to show a bijection between an increasing tree on [n+1] and a permutation on [n]. We start by writing our permutation in cycle notation so that the smallest node of each cycle appears at the beginning of the cycle representation.
Having this cycle representation, we start our tree with the 0-th node as the root of the increasing tree and the children of the root node are going to be the smallest node of each cycle. The remaining nodes in the cycle can be seen as a permutation themselves. Therefore, one can recursively determine the children of each node.



In the picture above, one can appreciate the graph construction from permutation 46328175*. At each level of the tree, one took the numbers in red as the labels of the nodes and the cycles of the permutations in black as their children.*

(2) *Since in the previous bijection the smallest element of each cycle corresponds to one of the roots in the root set of the increasing forest, the number of cycles in a permutation is equal to the number of components in the corresponding increasing forest. Therefore, the number of permutations with k cycles must be equal to the number of increasing forests with k components.*

136 *[2] This proof follows*
https://math.berkeley.edu/ mhaiman/math172-spring10/trees.pdf
Let $T(x)$ be the exponential generating function that enumerates rooted label trees:

$$T(x) = \sum_{n=1}^{\infty} n^{n-1}\frac{x^n}{n!}$$

Let $F(x)$ be the exponential generating function that enumerates functional digraphs:

$$F(x) = \sum_{n}^{\infty} n^n \frac{x^n}{n!}$$

Finally, let $P(x)$ be the exponential generating function that enumerates permutations:

$$P(x) = \sum_{n}^{\infty} n!\frac{x^n}{n!} = \frac{1}{1-x}$$

For this proof, we use the following claim about functional digraphs: every functional graph uniquely decomposes into disjoint rooted trees feeding into one or more disjoint cycles. Suppose the function corresponding to the functional digraph G_f is f. The key observation to prove the previous claim is that if we start at vertex x of G_f and follow the path $x, f(x), f^2(x)\cdots$, we'll eventually reach a vertex we already visited because our vertex set is finite. More details about the proof to this claim are given in the previously mentioned link. Since there is a bijection between a set of disjoint cycles and a permutation, the previous claim describes an equivalence between functional digraphs and the composition of (permutations $\circ$ rooted trees). By the composition principle of exponential generating functions (which can be found at https://math.berkeley.edu/ mhaiman/math172-spring10/exponential.pdf), this implies:

$$F(x) = (P \circ T)(x) = \frac{1}{1-T(x)}$$

Which is the desired expression.

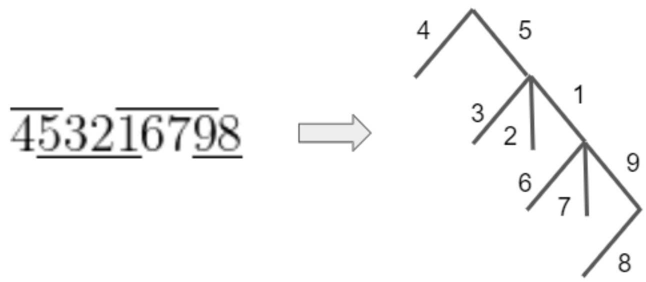
137 *[3] A solution to this problem can be found in I. M.Gessel and D.-L. Wang, /. Combinatorial Theory (A) 26 (1979), 308-313.*

138 *[3] Can be found in Chauve, Cedric, Serge Dulucq, and Andrew Rechnitzer. "Enumerating alternating trees." Journal of Combinatorial Theory, Series A 94, no. 1 (2001): 142-151.*

139 *[3-] Can be found in Postnikov, Alexander. "Intransitive trees." journal of combinatorial theory, Series A 79, no. 2 (1997): 360-366.*

141 *[2] The following proof is due to Postnikov and can be found at page 136 (exercise 5.41.k) of Stanley, Richard P. Enumerative Combinatorics. Volume 2. Cambridge, U.K.: Cambridge University Press, 1999.*
Consider a permutation p of $[n]$. We construct a tree such that the edges labelled i and j

have a common vertex if and only if they belong to the same monotone substring of p.
To illustrate this, consider the following permutation of [9] were the increasing substrings have an overline, and the decreasing ones have an underline:



Claim: The resulting graph is a tree.
The resulting graph is connected because the edges with labels in the same substring have a common vertex and can reach edges with labels in a consecutive substring through the label that belongs to both substrings.
The resulting graph has no cycles because if there was a path $e_1 e_2 \cdots e_k e_1$ *of edges, then there must be a substring* S_1 *such that* $\{e_1, e_2\} \in S_1$ *and a nonconsecutive substring* S_2 *such that* $\{e_k, e_1\} \in S_2$. *In this case, there would be 2 occurrences of* e_1 *in the permutation, which cannot happen.*

Claim: The resulting tree is an edge-labelled alternating tree.
Note that if the label i *of an edge belongs to only one monotone substring (like* 3 *in the example), then one of its ends is not incident to any other edges (one of its ends is a leave). These kind of edges cannot be the middle edge of any path of 3 edges.*
However, if the label i *of an edge belongs to two monotone consecutive substrings, then we have two cases:*

- *The left substring is increasing, and the right substring is decreasing. This is the case of* 9 *in the example. In this case, the label is the largest of all the labels in the two substrings. If we take an arbitrary path of 3 edges that contains the corresponding edge* i *in the middle, then the path must have alternating edges since all the edges incident to* i *are smaller than* i.
- *The left substring is decreasing, and the right substring is increasing. This is the case of* 1 *in the example. In this case the label is the smallest of all the labels in the two substrings. If we take an arbitrary path of 3 edges that contains the corresponding edge* i *in the middle, then the path must have alternating edges since all the edges incident to* i *are larger than* i.

Because of this, the resulting tree is and edge-labelled alternating tree.

Note that every tree can be obtained from 2 permutations: p and its reverse. Therefore, the number of edge-labelled alternating trees is $\frac{n!}{2}$.

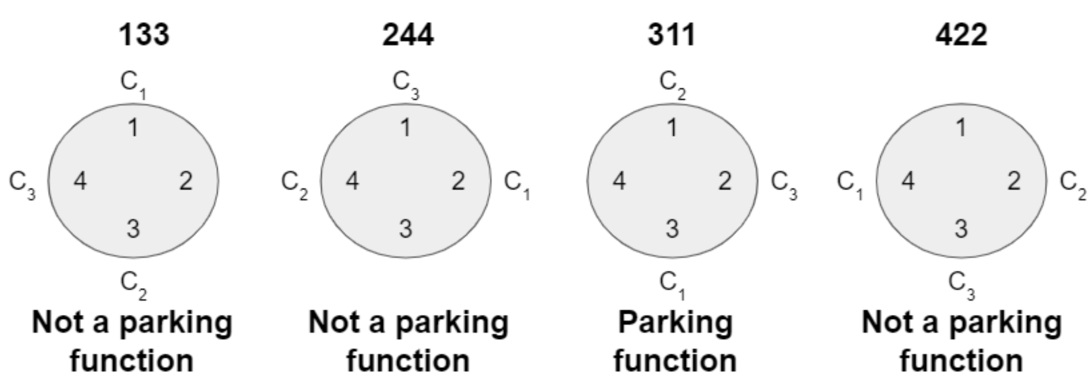
142 *[2+] Let's consider the set* S *of rooted spanning trees of* K_{mn} *such that the root belongs to* A. *Since* $|A| = m$, *then* $|S| = m \cdot c(K_{mn})$.
On the other hand, as in problem 128, we can represent the rooted tree as a directed graph

where the edges point towards the children. Since the root is in A, every single element in B has a parent in A. The number of ways the parents of the elements in B can be assigned is m^n.
Once those directed edges have been included to the graph, one has m connected components and their root set is B. To join them, one can successively add edges by picking an element in A to be the tail of the edge, and one of the remaining elements in the root set to be the head (keeping in mind that the head and the tail cannot belong to the same connected component). This gives us

$$\prod_{i=1}^{m-1} n(m-i) = n^{m-1}(m-1)!$$

Since these are ordered sequences of edges, the number of ways to choose the remaining $m-1$ edges regardless of the order is n^{m-1}.
Putting this together, we get that $S = m^n n^{m-1}$, which means that $c(K_{mn}) = m^{n-1}n^{m-1}$.

144 *[3-] A bijective solution to this problem is due to Pollak (1974) and can be found in Stanley, Richard P. "Parking functions." (1999). The solution defines a parking function in a different but equivalent way: Consider n cars who need to park in a line. The i-th car would prefer to park in the position a_i of the line, but if this position is occupied it would park in the next available one. The sequence $a_1a_2\cdots a_n$ is a parking function if all cars can park.*
With this definition, we consider an additional parking space $n+1$ and arrange the spaces in a circle. In a circle arrangement, cars can always park no matter the sequence of preferences. However, the sequence will correspond to a parking function if and only if the space $n+1$ is left empty.
If a sequence of preferences where C_i prefers a_i leads to a configuration where $n+1$ is empty (i.e. a parking function), then the sequence of preferences where C_i prefers a_i+j leads to a configuration where j is empty. Therefore, for each sequence of preferences corresponding to a parking function, there are n sequences that don't.
As an example, let $n = 3$ and consider the the sequence of preferences 133 in the circle configuration. If we add $i \bmod 4$ to each of its elements, we get the sequences 133, 244, 311, and 422, all of which leave empty a different position. Therefore, the only one that is a parking function is 311.



Since there are $(n+1)^n$ possible sequences of preferences, the number of parking functions must be $\frac{(n+1)^n}{n+1} = (n+1)^{n-1}$.

149 *[3] A proof to this problem can be found in J. Denes, Publ. Math. InstituteHungar. Acad. Sci. (= Magyar Tud. Akad. Mat. Kutato Int. Kozl) 4 (1959),63-71. This is a bijective proof that the number of ways to write some* n*-cycle as a product of* $n-1$ *transpositions is* $(n-1)!n^{n-2}$.

153 *[2-] See* [4], *Theorem 1.5.1.*

154 *[1+] See* [4], *Theorem 1.5.1.*

155 *[1+] See* [4], *Theorem 1.5.1.*

156 *[1+] We will provide a bijection to* Ballot Sequences *(i.e. sequence of* 1*'s and* -1*'s such that all partial sums are non-negative, Problem 164). Do a depth-first search, recording* 1 *when a left edge is first encountered, and recording* -1 *when a right edge is first encountered. Note also that when all endpoints are removed (together with the incident edges), we obtain the trees of item 155.*

157 *[2] See* [4], *Theorem 1.5.1.*

158 *[1+] In this solution, when we refer to a path we will mean a lattice path with steps* $(0,1)$ *or* $(1,0)$. *We will use the Principle of Inclusion-Exclusion. In this problem we are looking for:* p_n = *#Paths from (0, 0) to (n, n) never rising above the line y=x*
p_n *is equal to the number of paths from* $(0,0)$ *to* (n,n), *minus* p'_n, *the number of paths from* $(0,0)$ *to* (n,n) *that touch the line* $y=x+1$ *at least once. Now we will show that there is a bijection from* p'_n *to the number of paths from* $(0,0)$ *to* $(n-1,n+1)$. *To see this, starting with a path in* p'_n, *consider the first time that the path touches the line* $y=x+1$, *and invert all the remaining steps (up for right and vice versa). This is equivalent to a reflecting the remaining path around* $y=x+1$, *and the initial ending point,* (n,n), *turns into* $(n-1,n+1)$.
Convince yourself that this is a bijection. This means that $p'_n=\binom{2n}{n-1}$, *and that*

$$\begin{aligned}
p_n &= \binom{2n}{n} - \binom{2n}{n-1} \\
&= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{(n-1)!n!}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{(2n)!}{(n-1)!n!}\left(\frac{n+1-n}{n(n+1)}\right) = \frac{(2n)!}{(n-1)!n!}\left(\frac{1}{n(n+1)}\right) \\
&= \frac{(2n)!}{n(n-1)!n!}\left(\frac{1}{n+1}\right) = \frac{(2n)!}{n!n!}\left(\frac{1}{n+1}\right) = \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$

Figure 1 above shows a lattice path from $(0,0)$ *to* $(3,3)$ *that crosses* $y=x+1$. *The original path is the continuous line, the red dashed line is* $y=x+1$, *and the black dashed one is the reflection of the rest of the path. It is clear from the picture that the resulting path is a lattice path from* $(0,0)$ *to* $(3,3)$, *and that the process is reversible.*

159 *[1] See* [4], *Theorem 1.5.1.*

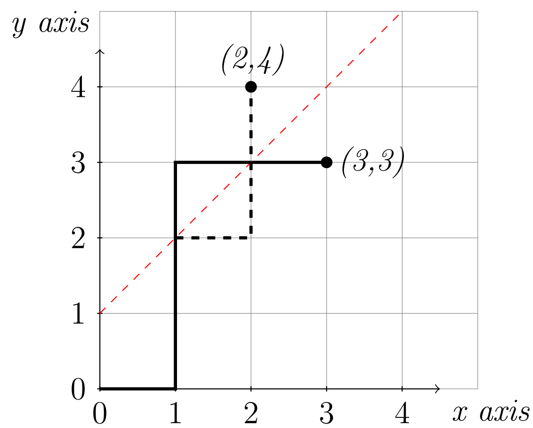


FIGURE 1

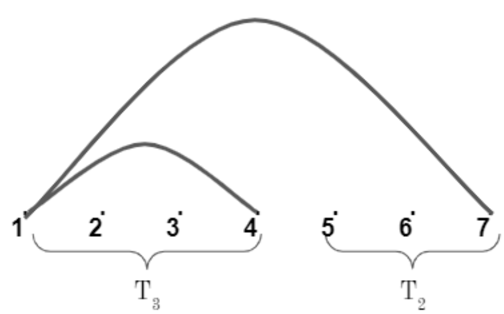
160 *[3-] See* [4], *page 66, Solution 57.*

161 *[2-] See* [4], *page 67, Solution 59.*

162 *[2] See* [4], *page 68, Solution 62. Here, we provide a different solution. Let T_n be the number of drawings with those properties in a plane with $n+1$ points. Let i be the largest node different from $n+1$ that is connected to 1 by an arc. We note the following:*

- *By (δ), the node i can only have arcs to points $< i$.*
- *By (γ), there cannot be arcs between points $\leq i$ and points $> i$.*

From this observation we can conclude, first, that in order for the resulting graph to be connected there must always be an arc between 1 and $n+1$. Second, that the nodes from 1 to i can only be connected among themselves, which by definition can be done in T_{i-1} ways. Similarly, the nodes from $i+1$ to $n+1$ con only be connected among themselves in T_{n-i} ways. Therefore, in this case we can obtain $T_{i-1}T_{n-i}$ valid drawings.



Adding all the possible cases, we get $T_n = \sum_{i=1}^{n} T_{i-1}T_{n-i}$, which is the recursive formula for Catalan numbers. Checking the smallest values of n by hand, one gets that $T_0 = 1$ and $T_1 = 1$, therefore $T_n = C_n$.

163 *[3-] See* [4], *page 68, Solution 63.*

164 *[3-] We will show a bijection with p_n:= the number of lattice paths from (0,0) to (n,n) that never cross $y = x$, which we know to be $c_n = \dfrac{1}{n+1}\binom{2n}{n}$ (from problem 158).*
We will code each path as a sequence of $\uparrow$ and $\rightarrow$, and notice that the partial count of up arrows is the y coordinate of a point in the path (similarly for the x coordinate and the right arrows). The condition that the path is never above the $y = x$ line means that $x \geq y \Rightarrow \# \rightarrow \geq \# \uparrow$. Now substitute the arrows such that a $\rightarrow$ is a $+1$, and a $\uparrow$ is a -1, and this adds the last link $\# \rightarrow \geq \# \uparrow \Rightarrow \#1 \geq \# -1 \Rightarrow$ each partial sum is non-negative. This means that each path in p_n transforms uniquely to a sequence of n $1's$ and n $-1's$, such that each partial sum is non-negative, and the process is reversible to a single lattice path.
Alternative solution: Let C_n be the number of elements in the set described by this problem. Let A be the set of all the sequences of n 1's and n -1's that don't satisfy the desired property. This is, such that there is a negative partial sum. Then, $|A| = \binom{2n}{n} - C_n$. It is enough to prove that $|A| = \binom{2n}{n+1} = \binom{2n}{n} - \frac{1}{n+1}\binom{2n}{n}$.
A set with $\binom{2n}{n+1}$ elements is the set of sequences of $n+1$ 1's and $n-1$ -1's. This is, the set of sequences of 1's and -1's that add up to 2. We are going to show a bijection between this set an A.
Take a sequence $(a_1, a_2, \cdots, a_{2n}) \in A$ and consider the first negative partial sum, which consists of the first k elements. Since this is the first negative partial sum, it must add up to -1. Then, we know that $\sum_{i=1}^{k} a_i = -1$ and $\sum_{i=k+1}^{2n} a_i = 1$.
Therefore if we flip the sign of the first i elements in this sequence, we are going to obtain a sequence that adds up to 2. We can retrieve the original sequence in A by finding the first partial sum that adds up to 1 and flipping the sign of its elements.
This is a popular solution equivalent to Andre's Reflection Method, a trick used to count the number of paths from problem 158.

165 *[1] Consider a lattice path P of the type of item 158. Let $a_i - 1$ be the area above the x-axis between $x = i - 1$ and $x = i$, and below P. This sets up a bijection as desired. Corrected from page 71, Solution 78* [4].

166 *[2] Translating the sequence into $b_i = i - a_i$ returns an object from Problem 165.*

167 *[1+] Take the first $n-1$ differences of each sequence in Problem 166, and this uniquely defines a sequence on this problem.*

168 *[2-] Do a depth-first search through a plane tree with n+1 vertices as in item 6. When a vertex is encountered for the first time, record one less than its number of children, except that the last vertex is ignored. This gives a bijection with Problem 157. See* [4]*, page 71, Solution 82.*

169 *[2-] Consider translating the sequence into a path in the following way:*

- *Start from* $(0, 0)$.
- *Each 1 is a step* $(1, 1)$.
- *Each -1 is a step* $(1, -1)$.
- *Once the sequence is over, add one extra step* $(1, 1)$.
- *Finally, add* $(1, -1)$ *steps until reaching* $(2n, 0)$.

This gives a bijection with Dyck paths (Problem 159). See [4]*, page 73, Solution 98.*

170 *[3-]See* [4]*, page 74, Solution 103.*

171 *[2+] Let* $\alpha = (\alpha_1, \dots \alpha_k), \beta = (\beta_1, \dots, \beta_k)$*. Define a Dyck path by going up* α_1 *steps, then down* β_1 *steps, then up* α_2 *steps, then down* β_2 *steps, etc. This gives a bijection with regular Dyck paths (Problem 159). See* [4]*, page 76, Solution 109.*

172 *[2] See* [4]*, page 78, Solution 113.*

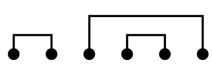
173 *[3-] See* [4]*, page 78, Solution 115.*

174 *[2] See* [4]*, page 79, Solution 116.*

175 *[2] See* [4]*, page 79, Solution 117.*

176 *[3-] Not in CN, search in other places.*

177 *[1+] There is a bijection to ballot sequences. Replace the left-hand endpoint of each arc with a* 1 *and the right-hand endpoint with* -1*. Let* M *be a non-crossing matching on* $2n$ *points. Suppose that we are given the set* S *of left endpoints of the arcs of* M*. We can recover* M *by scanning the elements of* S *from right to left, and attaching each element to the leftmost available point to its right. Also see* [4]*, page 67, Solution 61.*

Example: M = 
is the representation of the ballot sequence $1, -1, 1, 1, -1, -1$*, where* $S = \{1, 3, 4\}$.

178 *[2+] See* [4]*, page 68, Solution 64.*

179 *[2] See* [4]*, page 87, Solution 159.*

180 *[3-] Consider the following bijection with item 156. Label the vertices* $1, 2, \dots, 2n+1$ *of a tree in DFS left-first order. Make* i *and* j *to be in the same block if* j *is a right child of* i*. See* [4]*, page 88, Solution 163.*

181 *[3-] See* [4]*, page 88, Solution 164.*

182 *[3-] See* [4]*, page 92, Solution 180.*

183 *[2+] See* [4]*, page 93, Solution 187.*

184 *[2-] See* [4]*, page 95, Solution 164.*

185 *[3-] See* [4]*, page 73, Solution 92.*

186 *[3] See* [4]*, page 95, Solution 194.*

187 *[3] See* [4]*, page 96, Solution 197.*

188 *[3] See* [4]*, page 99, Solution 207.*

189 *[2+] Label each vertex by its degree. A saturated chain from the root to a vertex at level* $n-1$ *is thus labeled by a sequence* $(b_1, b_2, \ldots, b_n)$*. Set* $a_i = i + 2 - b_i$*. This sets up a bijection between level* $n-1$ *and the sequences* $(a_1, \ldots, a_n)$ *of item 165. Also see* [4]*, page 60, Solution 23.*

190 *[3-] See* [4]*, page 96, Solution 201.*

191 *[2+] Let* a_i *be the multiplicity of* $e_i - e_{i+1}$ *in the sum. The entire sum is uniquely determined by the sequence* $a_1,\ a_2, \ldots,\ a_{n-1}$*. Moreover, the sequences* $0, a_1, a_2, \ldots, a_{n-1}$ *that arise in this way coincide with those in item 166. Also see* [4]*, page 97, Solution 203.*

192 *[1+] There is a simple bijection with the binary trees* T *of item 155. The root of* T *corresponds to the rectangle containing the upper right-hand corner of the staircase. Remove this rectangle and we get two smaller staircase tilings (the one directly to the left might be empty), making the bijection obvious. Also see* [4]*, page 98, Solution 205.*

193 *[2+] Let*

$$(1, 1, ..., 1, -n) = \sum_{i=1}^{n-1} \left(a_i(e_i - e_{i+1}) + b_i(e_i - e_{n+1})\right) + a_n(e_n - e_{n+1})$$

as in item 191. Set $m_{ii} = a_i$ *and* $m_{in} = b_i$*. This uniquely determines the matrix* M *and sets up a bijection with item 191. Also see* [4]*, page 99, Solution 208.*

194 *[*] See P.Hajnal, G. Nagy, "A bijective proof of Shapiro's Catalan convolution", The Electronic Journal of Combinatorics (2014)*

196 *[2+] See A24 in* [4]*.*

197 *[3-] See exercise 6.36.a in* [6]*.*

199 *[*] See L.Grisales, R.Posada, S.Dimitrov, "Digraphs with exactly one Eulerian tour", to appear.*

200 *[1] We construct a bijection with the set of sequences of n 1's and n -1's that have no negative partial sums. For this, the numbers in the first row correspond to the positions occupied by 1's and the numbers in the second row correspond to the positions occupied by -1's.*
Note that if both rows are listed in increasing order, it is not a Standard Young Tableaux if and only if there is a column i in which the top number exceeds the bottom number, which happens if and only if the i-th -1 happens before the i-th 1 (i.e. there is a negative partial sum).
This is a solution by Brian Scott that can be found at

https://math.stackexchange.com/questions/1954309/whats-the-relation-between-standard-young-tableaux-and-catalan-number.

230 *[2+] Let S be the set of lattice paths of length 2n from $(0,0)$ to $(0,0)$ with steps $(0,\pm1)$ and $(\pm1,0)$. Let S' be the set of lattice paths of length 2n from $(0,0)$ to $(0,0)$ with steps $\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$ and $\left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{\sqrt{2}}\right)$. If we take, $s\in S$ and perform a clockwise rotation of $45°$, we can easily see that the resulting path s' belongs to S', which defines a bijection between S and S'.*
The number of paths in S' can be counted as follows: we first choose the first coordinate of all the steps. Since the paths end at $(0,0)$, there must be exactly n $\frac{1}{\sqrt{2}}$'s and exactly n $-\frac{1}{\sqrt{2}}$'s. This gives us $\binom{2n}{n}$ possible ways of choosing the first coordinate of all the steps. Since we can use a similar argument for the second coordinate, we get that the number of paths in S' is $\binom{2n}{n}^2$.

231 *[1] Any valid path has to visit either the coordinate (m,n-1), (m-1, n), or (m-1, n-1) right before arriving to (m,n). By definition, the number of paths from (0,0) to each of these coordinates is $f(m,n-1)$, $f(m-1,n)$, and $f(m-1,n-1)$ respectively. Adding those cases up, we obtain the desired equality.*

233 *[2] The following solution is very similar to the solution offered to problem 164.*
Let S be the set of sequences of m 1's and n -1's such that all the partial sums are positive. Let A be the set of $m-1$ 1's and n -1's. Then, $|B|=\binom{m+n-1}{n}$.
Let A' be the set of sequences in A such that all the partial sums are non-negative. There is a clear bijection between the lattice paths defined in this problem and the sequences of S. It's enough to transform any vector $(1,0)$ into a 1, any vector $(0,1)$ into a -1, and vice-versa.
Consider a sequence $s\in S$. It is clear that the first element is always going to be a 1. Therefore, a is uniquely defined by the remaining elements of the sequence, which form a sequence in A'. This implies $|S|=|A'|$.
We are going to demonstrate that

$$|A'|=\binom{m+n-1}{n}-\binom{m+n-1}{m}=|A|-\binom{m+n-1}{m}$$

It is enough to prove that the number of sequences in A such that there is a negative partial sum is $\binom{m+n-1}{m}$. Take such a sequence $x=(x_1,x_2,\cdots,x_{2n})$, and consider its first negative partial sum, which WLOG consists of the first k elements. Since this is the first negative partial sum it must add up to -1. Then, we know that $\sum_{i=1}^{k}x_i=-1$ and $\sum_{i=k+1}^{2n}x_i=m-n$. Therefore if we flip the sign of the first i elements in this sequence, we are going to obtain a sequence that adds up to $m-n+1$. We can retrieve the original sequence x by finding the first partial sum that adds up to 1 and flipping the sign of its elements. The number of sequences that add up to $m-n+1$ is $\binom{m+n-1}{m}$, as desired.

240 *[*] See Haupt, A.M., 2020. Bijective enumeration of rook walks. arXiv preprint arXiv:2007.01018.*

4. Additional problems

problem 13D from [7] ***(similar to p.5)*** *Let S be the set of all ordered $k-$tuples $(A_1, A_2, \ldots, A_k)$, such that $A_j \subseteq [n]$. Calculate $\sum_{\mathcal{A} \in S} \mid A_1 \cup A_2 \cup \ldots \cup A_k \mid$.*

Solution. *We can assume $n \geq 1$. The number of times the first element is counted in the wanted sum is the number of $n \times k$ 0-1 matrices with at least one 1 in the first row. This number is $(2^k - 1).2^{(n-1).k}$. Thus, due to summetry the answer is $n.(2^k - 1).2^{(n-1).k}$*

[1-] *Prove that $\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \ldots \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}$.*

Solution. *Consider a choice of 5 elements out $n+3$ and look at their positions $i_1 < i_2 < i_3 < i_4 < i_5$. The given equality is describing partitioning of the set of these choices by the position i_3. i_3 can be between 3 and n.*

[p66 EC2, ch.1, 2-] *Let $p_k(n)$ denote the number of partitions of n into k parts. Give a bijective proof that*

$$p_0(n) + p_1(n) + \ldots + p_k(n) = p_k(n+k)$$

Solution *Take an arbitrary partition μ of $n+k$ with k parts. If we remove 1 from each part (several of these parts could disappear if they were of size 1) then we get a partition of n. Apparantely, for different partitions of μ we get different partitions of n after performing the removal. Conversely, given an arbitrary partition σ of n*

[p68, EC2, ch.1, 2] *Let $n \geq 1$ and let $f(n)$ be the number of partitions of n such that for all k, the part k occurs at most k times. Let $g(n)$ be the number of partitions of n such that no part has the form $i(i+1)$, i.e., no parts equal to $2, 6, 12, 20, \ldots,$. Show that $f(n) = g(n)$.*

Solution *let $A = \{\mu \mid \mu$ - partition of n, $\forall k : k$ occurs at most k times $\}$ and $B = \{\mu \mid \mu$ - partition of n, $\forall i :$ no part $i(i+1)$ of $\mu\}$. Instead of bijection between A and B we'll show a bijection between $\bar{A}$ and $\bar{B}$. Let $\mu \in \bar{A}$. For each part of the kind $i(i+1)$ for some i, delete this part and add new $i+1$ parts of size i. As a result we have a partition without parts of kind $i(i+1)$. Example: $(6,3) \to (3,2,2,2)$. Notice that this map is injective since if $\mu_1 \to (3,2,2,2)$ for some $\mu_0 \neq \mu$, then μ_1 should not contain any of the 2's(2= 1.2), then μ_1 contains a 6 which transformed to $(2,2,2)$. The argument could be generalized to show injectivity. Conversely, given some $\sigma \in \bar{B}$ apply the following operation while it's possible - take $i+1$ parts of size i for some i, delete them and add one part of size $i(i+1)$.*

Acknowledgments

The first author is grateful for the help of prof. Janet Beissinger and the discussions that they had on several of the listed problems during the Spring semester of 2016!

References

[1] Babbage, C. (1819). Demonstration of a theorem relating to prime numbers. A. Constable.
[2] Stanley, R. P. (2013). Algebraic combinatorics. Springer, 20, 22.
[3] Stanley, R. P. (2009). Bijective proof problems. URL: `http://www-math. mit. edu/rstan/bij.pdf`.
[4] Stanley, R. P. (2015). Catalan numbers. Cambridge University Press.

[5] Stanley, R. P. (2011). Enumerative Combinatorics Volume 1 second edition. Cambridge studies in advanced mathematics.
[6] Stanley, R. P. (1999). Enumerative combinatorics. Vol. 2, volume 62 of. Cambridge Studies in Advanced Mathematics.
[7] Van Lint, J. H., Wilson, R. M., & Wilson, R. M. (2001). A course in combinatorics. Cambridge university press.
[8] Wolstenholme, J. (1862). On certain properties of prime numbers. QJ Math., 5, 35-39.