Chapter 1

What Is Combinatorics?

It would be surprising indeed if a reader of this book had never solved a combinatorial problem. Have you ever counted the number of games n teams would play if each team played every other team exactly once? Have you ever attempted to trace through a network without removing your pencil from the paper and without tracing any part of the network more than once? Have you ever counted the number of poker hands that are full houses in order to determine the odds against a full house? More recently, have you ever solved a Sudoku puzzle? These are all combinatorial problems. As these examples might suggest, combinatorics has its roots in mathematical recreations and games. Many problems that were studied in the past, either for amusement or for their aesthetic appeal, are today of great importance in pure and applied science. Today, combinatorics is an important branch of mathematics. One of the reasons for the tremendous growth of combinatorics has been the major impact that computers have had and continue to have in our society. Because of their increasing speed, computers have been able to solve large-scale problems that previously would not have been possible. But computers do not function independently. They need to be programmed to perform. The bases for these programs often are combinatorial algorithms for the solutions of problems. Analysis of these algorithms for efficiency with regard to running time and storage requirements demands more combinatorial thinking.

Another reason for the continued growth of combinatorics is its applicability to disciplines that previously had little serious contact with mathematics. Thus, we find that the ideas and techniques of combinatorics are being used not only in the traditional area of mathematical application, namely the physical sciences, but also in the social sciences, the biological sciences, information theory, and so on. In addition, combinatorics and combinatorial thinking have become more and more important in many mathematical disciplines.

Combinatorics is concerned with arrangements of the objects of a set into patterns satisfying specified rules. Two general types of problems occur repeatedly:

- *Existence of the arrangement.* If one wants to arrange the objects of a set so that certain conditions are fulfilled, it may not be at all obvious whether such an arrangement is possible. This is the most basic of questions. If the arrangement is not always possible, it is then appropriate to ask under what conditions, both necessary and sufficient, the desired arrangement can be achieved.

- *Enumeration or classification of the arrangements.* If a specified arrangement is possible, there may be several ways of achieving it. If so, one may want to count or to classify them into types.

If the number of arrangements for a particular problem is small, the arrangements can be listed. It is important to understand the distinction between listing all the arrangements and determining their number. Once the arrangements are listed, they can be counted by setting up a one-to-one correspondence between them and the set of integers $\{1, 2, 3, \ldots, n\}$ for some n. This is the way we count: one, two, three, ... However, we shall be concerned primarily with techniques for determining the number of arrangements of a particular type without first listing them. Of course the number of arrangements may be so large as to preclude listing them all.

Two other combinatorial problems often occur.

- *Study of a known arrangement.* After one has done the (possibly difficult) work of constructing an arrangement satisfying certain specified conditions, its properties and structure can then be investigated.

- *Construction of an optimal arrangement.* If more than one arrangement is possible, one may want to determine an arrangement that satisfies some optimality criterion—that is, to find a "best" or "optimal" arrangement in some prescribed sense.

Thus, a general description of combinatorics might be that *combinatorics is concerned with the existence, enumeration, analysis, and optimization of discrete structures.* In this book, *discrete* generally means "finite," although some discrete structures are infinite.

One of the principal tools of combinatorics for verifying discoveries is *mathematical induction.* Induction is a powerful procedure, and it is especially so in combinatorics. It is often easier to prove a stronger result than a weaker result with mathematical induction. Although it is necessary to verify more in the inductive step, the inductive hypothesis is stronger. Part of the art of mathematical induction is to find the right *balance of hypotheses and conclusions* to carry out the induction. We assume that the reader is familiar with induction; he or she will become more so as a result of working through this book.

The solutions of combinatorial problems can often be obtained using ad hoc arguments, possibly coupled with use of general theory. One cannot always fall back on application of formulas or known results. A typical solution of a combinatorial problem might encompass the following steps: (1) Set up a mathematical model, (2) study the model, (3) do some computation for small cases in order to develop some confidence and insight, and (4) use careful reasoning and ingenuity to finally obtain the solution of the problem. For counting problems, the inclusion–exclusion principle, the so-called pigeonhole principle, the methods of recurrence relations and generating functions, Burnside's theorem, and Pólya's counting formula are all examples of general principles and methods that we will consider in later chapters. Often, however, cleverness is required to see that a particular method or formula can be applied and how to apply. Thus, experience in solving combinatorial problems is very important. *The implication is that with combinatorics, as with mathematics in general, the more problems one solves, the more likely one is able to solve the next problem.*

We now consider a few introductory examples of combinatorial problems. They vary from relatively simple problems (but whose solution requires ingenuity) to problems whose solutions were a major achievement in combinatorics. Some of these problems will be considered in more detail in subsequent chapters.

1.1 Example: Perfect Covers of Chessboards

Consider an ordinary chessboard which is divided into 64 squares in 8 rows and 8 columns. Suppose there is available a supply of identically shaped dominoes, pieces which cover exactly two adjacent squares of the chessboard. Is it possible to arrange 32 dominoes on the chessboard so that no 2 dominoes overlap, every domino covers 2 squares, and all the squares of the chessboard are covered? We call such an arrangement a *perfect cover* or *tiling* of the chessboard by dominoes. This is an easy arrangement problem, and we can quickly construct many different perfect covers. It is difficult, but nonetheless possible, to count the number of different perfect covers. This number was found by Fischer[1] in 1961 to be $12,988,816 = 2^4 \times 17^2 \times 53^2$. The ordinary chessboard can be replaced by a more general chessboard divided into mn squares lying in m rows and n columns. A perfect cover need not exist now. Indeed, there is no perfect cover for the 3-by-3 board. For which values of m and n does the m-by-n chessboard have a perfect cover? It is not difficult to see that an m-by-n chessboard will have a perfect cover if and only if at least one of m and n is even or, equivalently, if and only if the number of squares of the chessboard is even. Fischer has derived general formulas involving trigonometric functions for the number of different perfect covers for the m-by-n chessboard. This problem is equivalent to a famous problem in molecular physics known as the *dimer problem.* It originated in the investigation of the absorption of diatomic atoms (dimers) on surfaces. The squares of the chessboard correspond to molecules, while the dominoes correspond to the dimers.

[1]M. E. Fischer, Statistical Mechanics of Dimers on a Plane Lattice, *Physical Review*, 124 (1961), 1664–1672.

Consider once again the 8-by-8 chessboard and, with a pair of scissors, cut out two diagonally opposite corner squares, leaving a total of 62 squares. Is it possible to arrange 31 dominoes to obtain a perfect cover of this "pruned" board? Although the pruned board is very close to being the 8-by-8 chessboard, which has over 12 million perfect covers, it has no perfect cover. The proof of this is an example of simple, but clever, combinatorial reasoning. In an ordinary 8-by-8 chessboard, usually the squares are alternately colored black and white, with 32 of the squares colored white and 32 of the squares colored black. If we cut out two diagonally opposite corner squares, we have removed two squares of the same color, say white. This leaves 32 black and 30 white squares. But each domino will cover one black and one white square, so that 31 nonoverlapping dominoes on the board cover 31 black and 31 white squares. We conclude that the pruned board has no perfect cover. The foregoing reasoning can be summarized by

$$31\,\fbox{B}\fbox{W} \neq 32\,\fbox{B} + 30\,\fbox{W}\,.$$

More generally, we can take an m-by-n chessboard whose squares are alternately colored black and white and arbitrarily cut out some squares, leaving a pruned board of some type or other. When does a pruned board have a perfect cover? For a perfect cover to exist, the pruned board must have an equal number of black and white squares. But this is not sufficient, as the example in Figure 1.1 indicates.

W	×	W	B	W
×	W	B	×	B
W	B	×	B	W
B	W	B	W	B

Figure 1.1

Thus, we ask: What are necessary and sufficient conditions for a pruned board to have a perfect cover? We will return to this problem in Chapter 9 and will obtain a complete solution. There, a practical formulation of this problem is given in terms of assigning applicants to jobs for which they qualify.

There is another way to generalize the problem of a perfect cover of an m-by-n board by dominoes. Let b be a positive integer. In place of dominoes we now consider 1-by-b pieces that consist of b 1-by-1 squares joined side by side in a consecutive manner. These pieces are called *b-ominoes.* and they can cover b consecutive squares in a row or b consecutive squares in a column. In Figure 1.2, a 5-omino is illustrated. A 2-omino is simply a domino. A 1-omino is also called a *monomino.*

Figure 1.2 A 5-omino

A *perfect cover* of an m-by-n board by b-ominoes is an arrangement of b-ominoes on the board so that (1) no two b-ominoes overlap, (2) every b-omino covers b squares of the board, and (3) all the squares of the board are covered. *When does an m-by-n board have a perfect cover by b-ominoes?* Since each square of the board is covered by exactly one b-omino, in order for there to be a perfect cover, b must be a factor of mn. Surely, a sufficient condition for the existence of a perfect cover is that b be a factor of m or b be a factor of n. For if b is a factor of m, we may perfectly cover the m-by-n board by arranging m/b b-ominoes in each of the n columns, while if b is a factor of n we may perfectly cover the board by arranging n/b b-ominoes in each of the m rows. Is this sufficient condition also necessary for there to be a perfect cover? Suppose for the moment that b is a prime number and that there is a perfect cover of the m-by-n board by b-ominoes. Then b is a factor of mn and, by a fundamental property of prime numbers, b is a factor of m or b is a factor of n. We conclude that, at least for the case of a prime number b, an m-by-n board can be perfectly covered by b-ominoes if and only if b is a factor of m or b is a factor of n.

In case b is not a prime number, we have to argue differently. So suppose we have the m-by-n board perfectly covered with b-ominoes. We want to show that either m or n has a remainder of 0 when divided by b. We divide m and n by b obtaining quotients p and q and remainders r and s, respectively:

$$\begin{aligned} m &= pb + r, \text{ where } & 0 \le r \le b-1, \\ n &= qb + s, \text{ where } & 0 \le s \le b-1. \end{aligned}$$

If $r = 0$, then b is a factor of m. If $s = 0$, then b is a factor of n. By interchanging the two dimensions of the board, if necessary, we may assume that $r \le s$. We then want to show that $r = 0$.

1	2	3	$\cdots$	$b-1$	b
b	1	2	$\cdots$	$b-2$	$b-1$
$b-1$	b	1	$\cdots$	$b-3$	$b-2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
2	3	4	$\cdots$	b	1

Figure 1.3 Coloring of a b-by-b board with b colors

We now generalize the alternate black-white coloring used in the case of dominoes ($b = 2$) to b colors. We choose b colors, which we label as 1, 2, ... , b. We color a b-by-b board in the manner indicated in Figure 1.3, and we extend this coloring to an

m-by-n board in the manner illustrated in Figure 1.4 for the case $m = 10$, $n = 11$, and $b = 4$.

Each b-omino of the perfect covering covers one square of each of the b colors. It follows that there must be the same number of squares of each color on the board. We consider the board to be divided into three parts: the upper pb-by-n part, the lower left r-by-qb part, and the lower right r-by-s part. (For the 10-by-11 board in Figure 1.4, we would have the upper 8-by-11 part, the 2-by-8 part in the lower left, and the 2-by-3 part in the lower right.) In the upper part, each color occurs p times in each column and hence pn times all together. In the lower left part, each color occurs q times in each row and hence rq times overall. Since each color occurs the same number of times on the whole board, it now follows that each color occurs the same number of times in the lower right r-by-s part.

1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2

Figure 1.4 Coloring of a 10-by-11 board with four colors

How many times does color 1 (and, hence, each color) occur in the r-by-s part? Since $r \leq s$, the nature of the coloring is such that color 1 occurs once in each row of the r-by-s part and hence r times in the r-by-s part. Let us now count the number of squares in the r-by-s part. On the one hand, there are rs squares; on the other hand, there are r squares of each of the b colors and so rb squares overall. Equating, we get $rs = rb$. If $r \neq 0$, we cancel to get $s = b$, contradicting $s \leq b - 1$. So $r = 0$, as desired. We summarize as follows:

An m-by-n board has a perfect cover by b-ominoes if and only if b is a factor of m or b is a factor of n.

A striking reformulation of the preceding statement is the following: Call a perfect cover *trivial* if all the b-ominoes are horizontal or all the b-ominoes are vertical. Then *an m-by-n board has a perfect cover by b-ominoes if and only if it has a trivial perfect cover.* Note that this does not mean that the only perfect covers are the trivial ones.

It does mean that if a perfect cover is possible, then a trivial perfect cover is also possible.

We conclude this section with a domino-covering problem with an added feature.

Consider a 4-by-4 chessboard that is perfectly covered with 8 dominoes. *Show that it is always possible to cut the board into two nonempty horizontal pieces or two nonempty vertical pieces without cutting through one of the 8 dominoes.* The horizontal or vertical line of such a cut is called a *fault line* of the perfect cover. Thus a horizontal fault line implies that the perfect cover of the 4-by-4 chessboard consists of a perfect cover of a k-by-4 board and a perfect cover of a $(4-k)$-by-4 board for some $k = 1, 2,$ or 3. Suppose there is a perfect cover of a 4-by-4 board such that none of the three horizontal lines and three vertical lines that cut the board into two nonempty pieces is a fault line. Let x_1, x_2, x_3 be, respectively, the number of dominoes that are cut by the horizontal lines (see Figure 1.5).

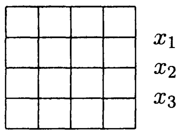


Figure 1.5

Because there is no fault line, each of x_1, x_2, and x_3 is positive. A horizontal domino covers two squares in a row, while a vertical domino covers one square in each of two rows. From these facts we conclude successively that x_1 is even, x_2 is even, and x_3 is even. Hence,

$$x_1 + x_2 + x_3 \geq 2 + 2 + 2 = 6,$$

and there are at least 6 vertical dominoes in the perfect cover. In a similar way, we conclude that there are at least 6 horizontal dominoes. Since $12 > 8$, we have a contradiction. Thus, it is impossible to cover perfectly a 4-by-4 board with dominoes without creating a fault line.

1.2 Example: Magic Squares

Among the oldest and most popular forms of mathematical recreations are *magic squares*, which have intrigued many important historical people. A magic square of order n is an n-by-n array constructed out of the integers $1, 2, 3, \ldots, n^2$ in such a way that the sum of the integers in each row, in each column, and in each of the two diagonals is the same number s. The number s is called the *magic sum* of the magic square. Examples of magic squares of orders 3 and 4 are

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}, \tag{1.1}$$

with magic sums 15 and 34, respectively. In medieval times there was a certain mysticism associated with magic squares; they were worn for protection against evils. Benjamin Franklin constructed many magic squares with additional properties.[2]

The sum of all the integers in a magic square of order n is

$$1 + 2 + 3 + \cdots + n^2 = \frac{n^2(n^2+1)}{2},$$

using the formula for the sum of numbers in an arithmetic progression (see Section 7.1). Since a magic square of order n has n rows each with magic sum s, we obtain the relation $ns = n^2(n^2+1)/2$. Thus, any two magic squares of order n have the same magic sum, namely,

$$s = \frac{n(n^2+1)}{2}.$$

The combinatorial problem is to determine for which values of n there is a magic square of order n and to find general methods of construction. It is not difficult to verify that there can be no magic square of order 2 (the magic sum would have to be 5). But, for all other values of n, a magic square of order n can be constructed. There are many special methods of construction. We describe here a method found by de la Loubère in the seventeenth century for constructing magic squares of order n when n is odd. First a 1 is placed in the middle square of the top row. The successive integers are then placed in their natural order along a diagonal line that slopes upward and to the right, with the following modifications:

(1) When the top row is reached, the next integer is put in the bottom row as if it came immediately above the top row.

(2) When the right-hand column is reached, the next integer is put in the left-hand column as if it had immediately succeeded the right-hand column.

(3) When a square that has already been filled is reached or when the top right-hand square is reached, the next integer is placed in the square immediately below the last square that was filled.

[2]See P. C. Pasles, The Lost Squares of Dr. Franklin: Ben Franklin's Missing squares and the Secret of the Magic Circle, *Amer. Math. Monthly*, 108 (2001), 489–511. Also see P. C. Pasles, *Benjamin Franklin's Numbers: An Unsung Mathematical Odyssey*, Princeton University Press, Princeton, NJ, 2008.

The magic square of order 3 in (1.1), as well as the magic square

$$\begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix}$$

of order 5, was constructed by using de la Loubère's method. Methods for constructing magic squares of even orders different from 2 and other methods for constructing magic squares of odd order can be found in a book by Rouse Ball.[3] Two of the magic squares of order 8 constructed by Franklin are as follows:

$$\begin{bmatrix} 52 & 61 & 4 & 13 & 20 & 29 & 36 & 45 \\ 14 & 3 & 62 & 51 & 46 & 35 & 30 & 19 \\ 53 & 60 & 5 & 12 & 21 & 28 & 37 & 44 \\ 11 & 6 & 59 & 54 & 43 & 38 & 27 & 22 \\ 55 & 58 & 7 & 10 & 23 & 26 & 39 & 42 \\ 9 & 8 & 57 & 56 & 41 & 40 & 25 & 24 \\ 50 & 63 & 2 & 15 & 18 & 31 & 34 & 47 \\ 16 & 1 & 64 & 49 & 48 & 33 & 32 & 17 \end{bmatrix}, \begin{bmatrix} 17 & 47 & 30 & 36 & 21 & 43 & 26 & 40 \\ 32 & 34 & 19 & 45 & 28 & 38 & 23 & 41 \\ 33 & 31 & 46 & 20 & 37 & 27 & 42 & 24 \\ 48 & 18 & 35 & 29 & 44 & 22 & 39 & 25 \\ 49 & 15 & 62 & 4 & 53 & 11 & 58 & 8 \\ 64 & 2 & 51 & 13 & 60 & 6 & 55 & 9 \\ 1 & 63 & 14 & 52 & 5 & 59 & 10 & 56 \\ 16 & 50 & 3 & 61 & 12 & 54 & 7 & 57 \end{bmatrix}.$$

These magic squares have some interesting properties. Can you see what they are?

Three-dimensional analogs of magic squares have been considered. A *magic cube* of order n is an n-by-n-by-n cubical array constructed out of the integers $1, 2, \ldots, n^3$ in such a way that the sum s of the integers in the n cells of each of the following straight lines is the same:

(1) lines parallel to an edge of the cube;

(2) the two diagonals of each plane cross section;

(3) the four space diagonals.

The number s is called the *magic sum* of the magic cube and has the value $(n^4 + n)/2$. We leave it as an easy exercise to show that there is no magic cube of order 2, and we verify that there is no magic cube of order 3.

Suppose that there is a magic cube of order 3. Its magic sum would then be 42. Consider any 3-by-3 plane cross section

$$\begin{bmatrix} a & b & c \\ x & y & z \\ d & e & f \end{bmatrix},$$

[3] W. W. Rouse Ball, *Mathematical Recreations and Essays*; revised by H. S. M. Coxeter. Macmillan, New York (1962), 193–221.

with numbers as shown. Since the cube is magic,

$$\begin{aligned} a+y+f &= 42 \\ b+y+e &= 42 \\ c+y+d &= 42 \\ a+b+c &= 42 \\ d+e+f &= 42. \end{aligned}$$

Subtracting the sum of the last two equations from the sum of the first three, we get $3y = 42$ and, hence, $y = 14$. But this means that 14 has to be the center of each plane cross section of the magic cube and, thus, would have to occupy seven different places. But it can occupy only one place, and we conclude that there is no magic cube of order 3. It is more difficult to show that there is no magic cube of order 4. A magic cube of order 8 is given in an article by Gardner.[4]

Although magic squares continue to interest mathematicians, we will not study them further in this book.

1.3 Example: The Four-Color Problem

Consider a map on a plane or on the surface of a sphere where the countries are connected regions.[5] To differentiate countries quickly, we must color them so that two countries that have a common boundary receive different colors (a corner does not count as a common boundary). What is the smallest number of colors necessary to guarantee that *every* map can be so colored? Until fairly recently, this was one of the famous unsolved problems in mathematics. Its appeal to the layperson is due to the fact that it can be simply stated and understood. More than any other mathematical problem, except possibly the well-known angle-trisection problem, the four-color problem has intrigued more amateur mathematicians, many of whom came up with faulty solutions. First posed by Francis Guthrie about 1850 when he was a graduate student, it has also stimulated a large body of mathematical research. Some maps require four colors. That's easy to see. An example is the map in Figure 1.6. Since each pair of the four countries of this map has a common boundary, it is clear that four colors are necessary to color the map. It was proven by Heawood[6] in 1890 that five colors are always enough to color any map. We give a proof of this fact in Chapter 12. It is not too difficult to show that it is impossible to have a map in the plane which

[4]M. Gardner, Mathematical Games, *Scientific American*, January (1976), 118–123.

[5]Thus, the state of Michigan would not be allowed as a country for such a map, unless we take into account that the upper and lower peninsulas of Michigan are connected by the Straits of Mackinac Bridge. Kentucky would also not be allowed, since its westernmost tip of Fulton County is completely surrounded by Missouri and Tennessee.

[6]P. J. Heawood, Map-Colour Theorems, *Quarterly J. Mathematics*, Oxford ser., 24 (1890), 332–338.

has five countries, every pair of which has a boundary in common. Such a map, if it had existed, would have required five colors. But not having five countries every two of which have a common boundary does not mean that four colors suffice. It might be that some map in the plane requires five colors for other more subtle reasons.

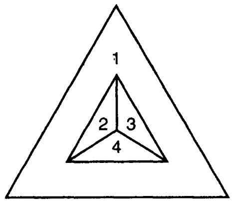


Figure 1.6

Now there are proofs that every planar map can be colored using only four colors, but they require extensive computer calculation.[7]

1.4 Example: The Problem of the 36 Officers

Given 36 officers of 6 ranks and from 6 regiments, can they be arranged in a 6-by-6 formation so that in each row and column there is one officer of each rank and one officer from each regiment? This problem, which was posed in the eighteenth century by the Swiss mathematician L. Euler as a problem in recreational mathematics, has important repercussions in statistics, especially in the design of experiments (see Chapter 10). An officer can be designated by an ordered pair (i, j), where i denotes his rank $(i = 1, 2, \ldots, 6)$ and j denotes his regiment $(j = 1, 2, \ldots, 6)$. Thus, the problem asks the following question:

> Can the 36 ordered pairs (i, j) $(i = 1, 2, \ldots, 6; j = 1, 2, \ldots, 6)$ be arranged in a 6-by-6 array so that in each row and each column the integers $1, 2, \ldots, 6$ occur in some order in the first positions and in some order in the second positions of the ordered pairs?

Such an array can be split into two 6-by-6 arrays, one corresponding to the first positions of the ordered pairs (the *rank array*) and the other to the second positions (the *regiment array*). Thus, the problem can be stated as follows:

> Do there exist two 6-by-6 arrays whose entries are taken from the integers $1, 2, \ldots, 6$ such that

[7]K. Appel and W. Haken, Every Planar Map is Four Colorable, *Bulletin of the American Mathematical Society*, 82 (1976), 711–712; K. Appel and W. Haken, *Every Planar Map is Four Colorable*, American Math. Society, Providence, RI (1989); and N. Robertson, D. P. Sanders, P. D. Seymour, and R. Thomas, The Four-Colour Theorem, *J. Combin. Theory Ser. B*, 70 (1997), 2–44.

(1) in each row and in each column of these arrays the integers $1, 2, \ldots, 6$ occur in some order, and

(2) when the two arrays are juxtaposed, all of the 36 ordered pairs (i, j) $(i = 1, 2, \ldots, 6; j = 1, 2, \ldots, 6)$ occur?

To make this concrete, suppose instead that there are 9 officers of 3 ranks and from 3 different regiments. Then a solution for the problem in this case is

$$\underset{\text{rank array}}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}}, \quad \underset{\text{regiment array}}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}} \quad \longrightarrow \quad \underset{\text{juxtaposed array}}{\begin{bmatrix} (1,1) & (2,2) & (3,3) \\ (3,2) & (1,3) & (2,1) \\ (2,3) & (3,1) & (1,2) \end{bmatrix}}. \tag{1.2}$$

The preceding rank and regiment arrays are examples of *Latin squares* of order 3; each of the integers 1, 2, and 3 occurs once in each row and once in each column. The following are Latin squares of orders 2 and 4:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}. \tag{1.3}$$

The two Latin squares of order 3 in (1.2) are called *orthogonal* because when they are juxtaposed, all of the 9 possible ordered pairs (i, j), with $i = 1, 2, 3$ and $j = 1, 2, 3$, result. We can thus rephrase Euler's question:

Do there exist two orthogonal Latin squares of order 6?

Euler investigated the more general problem of orthogonal Latin squares of order n. It is easy to see that there is no pair of orthogonal Latin squares of order 2, since, besides the Latin square of order 2 given in (1.3), the only other one is

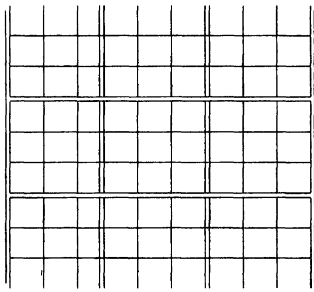
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

and these are not orthogonal. Euler showed how to construct a pair of orthogonal Latin squares of order n whenever n is odd or has 4 as a factor. Notice that this does not include $n = 6$. On the basis of many trials he concluded, but did not prove, that there is no pair of orthogonal Latin squares of order 6, and he conjectured that no such pair existed for any of integers $6, 10, 14, 18, \ldots, 4k + 2, \ldots$. By exhaustive enumeration, Tarry[8] in 1901 proved that Euler's conjecture was true for $n = 6$. Around 1960,

[8]G. Tarry, Le Problème de 36 officiers, *Compte Rendu de l'Association Française pour l'Avancement de Science Naturel*, 1 (1900), 122–123; 2 (1901), 170–203.

three mathematician-statisticians, R. C. Bose, E. T. Parker, and S. S. Shrikhande,[9] succeeded in proving that Euler's conjecture was false for all $n > 6$. That is, they showed how to construct a pair of orthogonal Latin squares of order n for every n of the form $4k+2$, $k = 2, 3, 4, \ldots$. This was a major achievement and put Euler's conjecture to rest. Later we shall explore how to construct orthogonal Latin squares using finite number systems called finite fields and how they can be applied in *experimental design.*

As a concluding remark to this section, we observe that in the number placement puzzle called *Sudoku*, which became an international success in 2005, one is asked to construct a special Latin square of order 9 that has been partitioned into nine 3-by-3 squares as follows:



In each Sudoku puzzle, some of the entries of a 9-by-9 square have been filled in such a way that there is a unique and logical way to complete it to a Latin square of order 9 with the additional constraint that each of the nine 3-by-3 squares contains the integers $1, 2, 3, 4, 5, 6, 7, 8, 9$. Thus each of the nine rows, columns, and 3-by-3 squares is to contain one each of the numbers $1, 2, \ldots, 9$. The level of difficulty of a Sudoku puzzle depends on the depth of the logic needed to determine how to fill the empty boxes and in what order.

An example of a Sudoku puzzle is

3		5				2		7
			7		3			
	4	6				5	8	
	3		1		9		6	
			2		7			
	8		4		5		9	
	2	1				6	3	
			8		6			
6		4				8		1

[9]R. C. Bose, E. T. Parker and S. S. Shrikhande, Further Results on the Construction of Mutually Orthogonal Latin squares and the Falsity of Euler's conjecture, *Canadian Journal of Mathematics*, 12 (1960), 189–203.

whose solution is

3	9	5	6	4	8	2	1	7
2	1	8	7	5	3	9	4	6
7	4	6	9	2	1	5	8	3
5	3	2	1	8	9	7	6	4
4	6	9	2	3	7	1	5	8
1	8	7	4	6	5	3	9	2
8	2	1	5	7	4	6	3	9
9	7	3	8	1	6	4	2	5
6	5	4	3	9	2	8	7	1

The solution to a Sudoku puzzle is an instance of a Latin square called *a gerechte design*, where an n-by-n square is partitioned into n regions each containing n squares and each of the integers $1, 2, \ldots, n$ occurs once in each row and columns (so we get a Latin square) and once in each of the n regions.[10]

We give a simple example of a gerechte design coming from a partitioning of a 4-by-4 square into four L-shaped regions containing four squares each. We use the symbols ♠, ♢, ♣, and ♡ to denote the different regions, as shown below.

♠	♠	♠	♢
♠	♢	♢	♢
♣	♡	♡	♡
♣	♣	♣	♡

$\longrightarrow$

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

1.5 Example: Shortest-Route Problem

Consider a system of streets and intersections. A person wishes to travel from one intersection A to another intersection B. In general, there are many available routes from A to B. The problem is to determine a route for which the distance traveled is as small as possible, a *shortest route.* This is an example of a *combinatorial optimization* problem. One possible way to solve this problem is to list in a systematic way all possible routes from A to B. It is not necessary to travel over any street more than once; thus, there is only a finite number of such routes. Then compute the distance traveled for each and select a shortest route. This is not a very efficient procedure and, when the system is large, the amount of work may be too great to permit a solution in a reasonable amount of time. What is needed is an algorithm for determining a shortest route in which the work involved in carrying out the algorithm does not increase too rapidly as the system increases in size. In other words, the amount of work should be bounded by a polynomial function (as opposed to, say, an exponential function) of the size of the problem. In Section 11.7 we describe such an algorithm.

[10]R. A. Bailey, P. J. Cameron, and R. Connelly, Sudoku, Gerechte Designs, Resolutions, Affine Spaces, Spreads, Reguli, and Hamming Codes, *Amer. Math. Monthly*, 115 (2008), 383–404.

This algorithm will actually find a shortest route from A to every other intersection in the system.

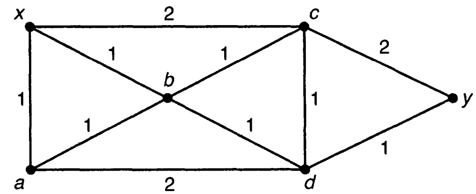


Figure 1.7

The problem of finding a shortest route between two intersections can be viewed abstractly. Let V be a finite set of objects called *vertices* (which correspond to the intersections and the ends of dead-end streets), and let E be a set of unordered pairs of vertices called *edges* (which correspond to the streets). Thus, some pairs of vertices are joined by edges, while others are not. The pair (V, E) is called a *graph*. A *walk* in the graph joining vertices x and y is a sequence of vertices such that the first vertex is x and the last vertex is y, and any two consecutive vertices are joined by an edge. Now associate with each edge a nonnegative real number, the *length* of the edge. The *length of a walk* is the sum of the lengths of the edges that join consecutive vertices of the walk. Given two vertices x and y, the shortest-route problem is to find a walk from x to y that has the smallest length. In the graph depicted in Figure 1.7, there are 6 vertices and 10 edges. The numbers on the edges denote their lengths. One walk joining x and y is x, a, b, d, y, and it has length 4. Another is x, b, d, y, and it has length 3. It is not difficult to see that the latter walk gives a shortest route joining x and y.

A graph is an example of a discrete structure which has been and continues to be extensively studied in combinatorics. The generality of the notion allows for its wide applicability in such diverse fields as psychology, sociology, chemistry, genetics, and communications science. Thus, the vertices of a graph might correspond to people, with two vertices joined by an edge if the corresponding people distrust each other; or the vertices might represent atoms, and the edges represent the bonds between atoms. You can probably imagine other ways in which graphs can be used to model phenomena. Some important concepts and properties of graphs are studied in Chapters 9, 11, and 12.

1.6 Example: Mutually Overlapping Circles

Consider n mutually overlapping circles $\gamma_1, \gamma_2, \ldots, \gamma_n$ in general position in the plane. By *mutually overlapping* we mean that each pair of the circles intersects in two distinct points (thus nonintersecting or tangent circles are not allowed). By *general position*, we mean that there do not exist three circles with a common point.[11] The n circles create a number of regions in the plane. The problem is to determine how many regions are so created.

Let h_n equal the number of regions created. We easily compute that $h_1 = 2$ (the inside and outside of the circle γ_1), $h_2 = 4$ (the usual Venn diagram for two sets), and $h_3 = 8$ (the usual Venn diagram for three sets). Since the numbers seem to be doubling, it is tempting now to think that $h_4 = 16$. However, a picture quickly reveals that $h_4 = 14$ (see Figure 1.8).

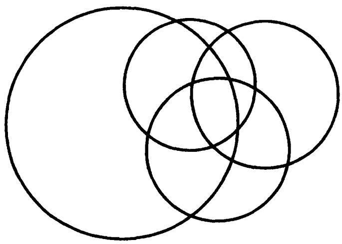


Figure 1.8 Four mutually overlapping circles in general position

One way to solve counting problems of this sort is to try to determine the change in the number of regions that occurs when we go from $n-1$ circles $\gamma_1, \ldots, \gamma_{n-1}$ to n circles $\gamma_1, \ldots, \gamma_{n-1}, \gamma_n$. In more formal language, we try to determine a recurrence relation for h_n; that is, express h_n in terms of previous values.

So assume that $n \geq 2$ and that the $n-1$ mutually overlapping circles $\gamma_1, \ldots, \gamma_{n-1}$ have been drawn in the plane in general position creating h_{n-1} regions. Then put in the nth circle γ_n so that there are now n mutually overlapping circles in general position. Each of the first $n-1$ circles intersects the nth circle γ_n in two points, and since the circles are in general position we obtain $2(n-1)$ distinct points $P_1, P_2, \ldots, P_{2(n-1)}$. These $2(n-1)$ points divide γ_n into $2(n-1)$ arcs: the arc between P_1 and P_2, the arc between P_2 and P_3, ... , the arc between $P_{2(n-1)-1}$ and $P_{2(n-1)}$, and the arc between $P_{2(n-1)}$ and P_1. Each of these $2(n-1)$ arcs divides a region formed by the first $n-1$ circles $\gamma_1, \ldots, \gamma_{n-1}$ into two, creating $2(n-1)$ more regions. Thus, h_n satisfies the relation

$$h_n = h_{n-1} + 2(n-1), \qquad (n \geq 2). \tag{1.4}$$

We can use the recurrence relation (1.4) to obtain a formula for h_n in terms of the parameter n. By iterating (1.4),[12] we obtain

$$h_n = h_{n-1} + 2(n-1)$$

[11] It is not necessary that the "circles" be round. Closed convex curves are sufficient.

[12] That is, applying (1.4) over and over again until finally we get to h_1 which we know to be 2.

$$\begin{aligned}
h_n &= h_{n-2} + 2(n-2) + 2(n-1) \\
h_n &= h_{n-3} + 2(n-3) + 2(n-2) + 2(n-1) \\
&\vdots \\
h_n &= h_1 + 2(1) + 2(2) + \cdots + 2(n-2) + 2(n-1).
\end{aligned}$$

Since $h_1 = 2$, and $1 + 2 + \cdots + (n-1) = n(n-1)/2$, we get

$$h_n = 2 + 2 \cdot \frac{n(n-1)}{2} = n^2 - n + 2, \qquad (n \geq 2).$$

This formula is also valid for $n = 1$, since $h_1 = 2$. A formal proof of this formula can now be given using mathematical induction.

1.7 Example: The Game of Nim

We close this introductory chapter by returning to the roots of combinatorics in recreational mathematics and investigating the ancient game of Nim.[13] Its solution depends on *parity*, an important problem-solving concept in combinatorics. We used a simple parity argument in investigating perfect covers of chessboards when we showed that a board had to have an even number of squares to have a perfect cover with dominoes.

Nim is a game played by two players with heaps of coins (or stones or beans). Suppose that there are $k \geq 1$ heaps of coins that contain, respectively, $n_1, n_2, \ldots, n_k$ coins. The *object* of the game is to select the last coin. The *rules* of the game are as follows:

(1) The players alternate turns (let us call the player who makes the first move I and then call the other player II).

(2) Each player, when it is his or her turn, selects one of the heaps and removes at least one of the coins from the selected heap. (The player may take all of the coins from the selected heap, thereby leaving an empty heap, which is now "out of play.")

The game ends when all the heaps are empty. The last player to make a move—that is, the player who takes the last coin(s)—is the *winner*.

The variables in this game are the number k of heaps and the numbers $n_1, n_2, \ldots, n_k$ of coins in the heaps. The combinatorial problem is to determine whether the first or second player wins[14] and how that player should move in order to guarantee a win—a *winning strategy*.

---

[13] *Nim* derives from the German *Nimm!*, meaning *Take!*.

[14] With intelligent play.

To develop some understanding of Nim, we consider some special cases.[15] If there is initially only one heap, then player I wins by removing all the coins. Now suppose that there are $k = 2$ heaps, with n_1 and n_2 coins, respectively. Whether or not player I can win depends not on the actual values of n_1 and n_2 but on whether or not they are equal. Suppose that $n_1 \neq n_2$. Player I can remove enough coins from the larger heap in order to leave two heaps of equal size for player II. Now player I, when it is her turn, can mimic player II's moves. Thus if player II takes c coins from one of the heaps, then player I takes the same number c of coins from the other heap. Such a strategy guarantees a win for player I. If $n_1 = n_2$, then player II can win by mimicking player I's moves. Thus, we have completely solved 2-heap Nim. An example of play in the 2-heap game of Nim with heaps of sizes 8 and 5, respectively, is

$$8,5 \xrightarrow{\text{I}} 5,5 \xrightarrow{\text{II}} 5,2 \xrightarrow{\text{I}} 2,2 \xrightarrow{\text{II}} 0,2 \xrightarrow{\text{I}} 0,0.$$

The preceding idea in solving 2-heap Nim, namely, moving in such a way as to leave two equal heaps, can be generalized to any number k of heaps. The insight one needs is provided by the concept of the base 2 numeral of an integer. Recall that each positive integer n can be expressed as a base 2 numeral by repeatedly removing the largest power of 2 which does not exceed the number. For instance, to express the decimal number 57 in base 2, we observe that

$$\begin{array}{ll} 2^5 \leq 57 < 2^6, & 57 - 2^5 = 25 \\ 2^4 \leq 25 < 2^5, & 25 - 2^4 = 9 \\ 2^3 \leq 9 < 2^4, & 9 - 2^3 = 1 \\ 2^0 \leq 1 < 2^1, & 1 - 2^0 = 0. \end{array}$$

Thus,

$$57 = 2^5 + 2^4 + 2^3 + 2^0,$$

and the base 2 numeral for 57 is

$$111001.$$

Each digit in a base 2 numeral is either 0 or 1. The digit in the ith position, the one corresponding to 2^i, is called the ith bit[16] ($i \geq 0$). We can think of each heap of coins as consisting of *subheaps* of powers of 2, according to its base numeral. Thus a heap of size 53 consists of subheaps of sizes $2^5, 2^4, 2^2$, and 2^0. In the case of 2-heap Nim, the total number of subheaps of each size is either 0, 1, or 2. There is exactly one subheap of a particular size if and only if the two heaps have different sizes. Put another way, the total number of subheaps of each size is even if and only if the two heaps have the same size—that is, if and only if player II can win the Nim game.

[15]This is an important principle to follow in general: Consider small or special cases to develop understanding and intuition. Then try to extend your ideas to solve the problem in general.

[16]The word *bit* is short for *bi*nary digi*t*.

Now consider a general Nim game with heaps of sizes $n_1, n_2, \ldots, n_k$. Express each of the numbers n_i as base 2 numerals:

$$\begin{aligned} n_1 &= a_s \cdots a_1 a_0 \\ n_2 &= b_s \cdots b_1 b_0 \\ &\cdots \\ n_k &= e_s \cdots e_1 e_0. \end{aligned}$$

(By including leading 0s, we can assume that all of the heap sizes have base 2 numerals with the same number of digits.) We call a Nim game *balanced*, provided that the number of subheaps of each size is even. Thus, a Nim game is balanced if and only if

$$\begin{gathered} a_s + b_s + \cdots + e_s \text{ is even,} \\ \vdots \\ a_i + b_i + \cdots + e_i \text{ is even,} \\ \vdots \\ a_0 + b_0 + \cdots + e_0 \text{ is even.} \end{gathered}$$

A Nim game that is not balanced is called *unbalanced.* We say that the ith bit is *balanced* provided that the sum $a_i + b_i + \cdots + e_i$ is even, and is *unbalanced* otherwise. Thus, a balanced game is one in which all bits are balanced, while an unbalanced game is one in which there is at least one unbalanced bit.

We then have the following:

> Player I can win in unbalanced Nim games, and player II can win in balanced Nim games.

To see this, we generalize the strategies used in 2-heap Nim. Suppose the Nim game is unbalanced. Let the largest unbalanced bit be the jth bit. Then player I moves in such a way as to leave a balanced game for player II. She does this by selecting a heap whose jth bit is 1 and removing a number of coins from it so that the resulting game is balanced (see also Exercise 32). No matter what player II does, she leaves for player I an unbalanced game again, and player I once again balances it. Continuing like this ensures player I a win. If the game starts out balanced, then player I's first move unbalances it, and now player II adopts the strategy of balancing the game whenever it is her move.

For example, consider a 4-heap Nim game with heaps of sizes 7, 9, 12, and 15. The base 2 numerals for these heap sizes are, respectively, 0111, 1001, 1100, and 1111. In terms of subheaps of powers of 2, we have:

	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
Heap of size 7	0	1	1	1
Heap of size 9	1	0	0	1
Heap of size 12	1	1	0	0
Heap of size 15	1	1	1	1

This game is unbalanced with the 3rd, 2nd and 0th bits unbalanced. Player I can remove 11 coins from the pile of size 12, leaving 1 coin. Since the base 2 numeral of 1 is 0001, the game is now balanced. Alternatively, player I can remove 5 coins from the pile of size 9, leaving 4 coins, or player I can remove 13 coins from the pile of size 15, leaving 2 coins.

1.8 Exercises

1. Show that an m-by-n chessboard has a perfect cover by dominoes if and only if at least one of m and n is even.

2. Consider an m-by-n chessboard with m and n both odd. To fix the notation, suppose that the square in the upper left-hand corner is colored white. Show that if a white square is cut out anywhere on the board, the resulting pruned board has a perfect cover by dominoes.

3. Imagine a prison consisting of 64 cells arranged like the squares of an 8-by-8 chessboard. There are doors between all adjoining cells. A prisoner in one of the corner cells is told that he will be released, provided he can get into the diagonally opposite corner cell after passing through every other cell exactly once. Can the prisoner obtain his freedom?

4. (a) Let $f(n)$ count the number of different perfect covers of a 2-by-n chessboard by dominoes. Evaluate $f(1), f(2), f(3), f(4)$, and $f(5)$. Try to find (and verify) a simple relation that the counting function f satisfies. Use this relation to compute $f(12)$.

   (b) * Let $g(n)$ be the number of different perfect covers of a 3-by-n chessboard by dominoes. Evaluate $g(1), g(2), \ldots, g(6)$.

5. Find the number of different perfect covers of a 3-by-4 chessboard by dominoes.

6. Consider the following three-dimensional version of the chessboard problem: A *three-dimensional domino* is defined to be the geometric figure that results when two cubes, one unit on an edge, are joined along a face. Show that it is possible to construct a cube n units on an edge from dominoes if and only if n is even. If n is odd, is it possible to construct a cube n units on an edge with a 1-by-1 hole in the middle? (*Hint*: Think of a cube n units on an edge as being composed of n^3 cubes, one unit on an edge. Color the cubes alternately black and white.)

7. Let a and b be positive integers with a a factor of b. Show that an m-by-n board has a perfect cover by a-by-b pieces if and only if a is a factor of both m and n and b is a factor of either m or n. (*Hint*: Partition the a-by-b pieces into a 1-by-b pieces.)

8. Use Exercise 7 to conclude that when a is a factor of b, an m-by-n board has a perfect cover by a-by-b pieces if and only if it has a trivial perfect cover in which all the pieces are oriented the same way.

9. Show that the conclusion of Exercise 8 need not hold when a is not a factor of b.

10. Verify that there is no magic square of order 2.

11. Use de la Loubère's method to construct a magic square of order 7.

12. Use de la Loubère's method to construct a magic square of order 9.

13. Construct a magic square of order 6.

14. Show that a magic square of order 3 must have a 5 in the middle position. Deduce that there are exactly 8 magic squares of order 3.

15. Can the following partial square be completed to obtain a magic square of order 4?

$$\begin{bmatrix} 2 & 3 & & \\ 4 & & & \\ & & & \\ & & & \end{bmatrix}$$

16. Show that the result of replacing every integer a in a magic square of order n with $n^2 + 1 - a$ is a magic square of order n.

17. Let n be a positive integer divisible by 4, say $n = 4m$. Consider the following construction of an n-by-n array:

    (1) Proceeding from left to right and from first row to nth row, fill in the places of the array with the integers $1, 2, \ldots, n^2$ in order.

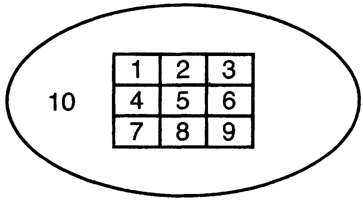
    (2) Partition the resulting square array into m^2 4-by-4 smaller arrays. Replace each number a on the two diagonals of each of the 4-by-4 arrays with its "complement" $n^2 + 1 - a$.

    Verify that this construction produces a magic square of order n when $n = 4$ and $n = 8$. (Actually it produces a magic square for each n divisible by 4.)

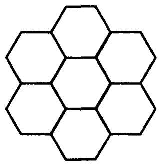
18. Show that there is no magic cube of order 2.

19. * Show that there is no magic cube of order 4.

20. Show that the following map of 10 countries $\{1, 2, \ldots, 10\}$ can be colored with three but no fewer colors. If the colors used are red, white, and blue, determine the number of different colorings.



21. (a) Does there exist a *magic hexagon* of order 2? That is, is it possible to arrange the numbers $1, 2, \ldots, 7$ in the following hexagonal array so that all of the nine "line" sums (the sum of the numbers in the hexagonal boxes penetrated by a line through midpoints of opposite sides) are the same?



(b) * Construct a magic hexagon of order 3; that is, arrange the integers $1, 2, \ldots, 19$ in a hexagonal array (three integers on a side) in such a way that all of the fifteen "line" sums are the same (namely, 38).

22. Construct a pair of orthogonal Latin squares of order 4.

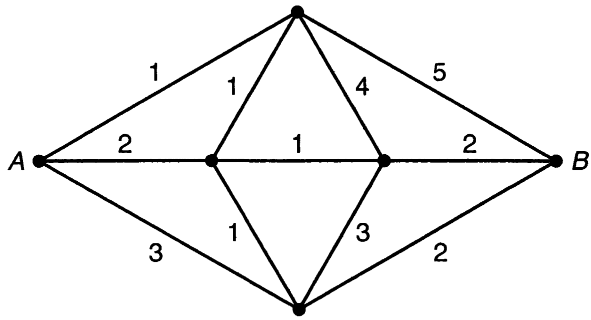
23. Construct Latin squares of orders 5 and 6.

24. Find a general method for constructing a Latin square of order n.

25. A 6-by-6 chessboard is perfectly covered with 18 dominoes. Prove that it is possible to cut it either horizontally or vertically into two nonempty pieces without cutting through a domino; that is, prove that there must be a fault line.

26. Construct a perfect cover of an 8-by-8 chessboard with dominoes having no fault-line.

27. Determine all shortest routes from A to B in the system of intersections and streets (graph) in the following diagram. The numbers on the streets represent the lengths of the streets measured in terms of some unit.



28. Consider 3-heap Nim with heaps of sizes 1, 2, and 4. Show that this game is unbalanced and determine a first move for player I.

29. Is 4-heap Nim with heaps of sizes 22, 19, 14, and 11 balanced or unbalanced? Player I's first move is to remove 6 coins from the heap of size 19. What should player II's first move be?

30. Consider 5-heap Nim with heaps of sizes 10, 20, 30, 40, and 50. Is this game balanced? Determine a first move for player I.

31. Show that player I can always win a Nim game in which the number of heaps with an odd number of coins is odd.

32. Show that in an unbalanced game of Nim in which the largest unbalanced bit is the jth bit, player I can always balance the game by removing coins from any heap the base 2 numeral of whose number has a 1 in the jth bit.

33. Suppose we change the object of Nim so that the player who takes the last coin loses (the *misère* version). Show that the following is a winning strategy: Play as in ordinary Nim until all but exactly one heap contains a single coin. Then remove either all or all but one of the coins of the exceptional heap so as to leave an *odd* number of heaps of size 1.

34. A game is played between two players, alternating turns as follows: The game starts with an empty pile. When it is his turn, a player may add either 1, 2, 3, or 4 coins to the pile. The person who adds the 100th coin to the pile is the winner. Determine whether it is the first or second player who can guarantee a win in this game. What is the winning strategy?

35. Suppose that in Exercise 34, the player who adds the 100th coin loses. Now who wins, and how?

36. Eight people are at a party and pair off to form four teams of two. In how many ways can this be done? (This is sort of an "unstructured" domino-covering problem.)

37. A Latin square of order n is *idempotent* provided the integers $\{1, 2, \ldots, n\}$ occur in the diagonal positions $(1,1), (2,2), \ldots, (n,n)$ in the order $1, 2, \ldots, n$, and is *symmetric* provided the integer in position (i,j) equals the integer in position (j,i) whenever $i \neq j$. There is no symmetric, idempotent Latin square of order 2. Construct a symmetric, idempotent Latin square of order 3. Show that there is no symmetric, idempotent Latin square of order 4. What about order n in general, where n is even?

38. Take any set of $2n$ points in the plane with no three collinear, and then arbitrarily color each point red or blue. Prove that it is always possible to pair up the red points with the blue points by drawing line segments connecting them so that no two of the line segments intersect.

39. Consider an n-by-n board and L-tetrominoes (4 squares joined in the shape of an L). Show that if there is a perfect cover of the n-by-n board with L-tetrominoes, then n is divisible by 4. What about m-by-n-boards?

40. Solve the following Sudoku puzzle,

			5					6
		8						7
7	5			6	4			
	3	6		8		2	4	5
	2		3		9		6	
5	1	7		2		8	3	
			2	4			7	8
4						3		
1					3			

41. Solve the following Sudoku puzzle,

7			1	5	4			8
2		5	9		8	1		6
		6	7		3	4		
	3						2	
		7	2		9	6		
8		3	4		2	9		5
5			8	7	6			2

42. Let S_n denote the staircase board with $1+2+\cdots+n = n(n+1)/2$ squares. For example, S_4 is

	×	×	×
		×	×
			×

Prove that S_n does not have a perfect cover with dominoes for any $n \geq 1$.

43. Consider a block of wood in the shape of a cube, 3 feet on an edge. It is desired to cut the cube into 27 smaller cubes, 1 foot on an edge. One way to do this is to make 6 cuts, 2 in each direction, while keeping the cube in one block. Is it possible to use fewer cuts if the pieces can be rearranged between cuts?

44. Show how to cut a cube, 3 feet on an edge, into 27 cubes, 1 foot on an edge, using exactly 6 cuts but making a nontrivial rearrangement of the pieces between two of the cuts.