Chapter 3

The Pigeonhole Principle

We consider in this chapter an important, but elementary, combinatorial principle that can be used to solve a variety of interesting problems, often with surprising conclusions. This principle is known under a variety of names, the most common of which are the *pigeonhole principle*, the *Dirichlet drawer principle*, and the *shoebox principle*.[1] Formulated as a principle about pigeonholes, it says roughly that if a lot of pigeons fly into not too many pigeonholes, then at least one pigeonhole will be occupied by two or more pigeons. A more precise statement is given below.

3.1 Pigeonhole Principle: Simple Form

The simplest form of the pigeonhole principle is the following fairly obvious assertion.

Theorem 3.1.1 *If $n+1$ objects are distributed into n boxes, then at least one box contains two or more of the objects.*

Proof. The proof is by contradiction. If each of the n boxes contains at most one of the objects, then the total number of objects is at most $1 + 1 + \cdots + 1 (n \text{ 1s}) = n$. Since we distribute $n+1$ objects, some box contains at least two of the objects. □

Notice that neither the pigeonhole principle nor its proof gives any help in finding a box that contains two or more of the objects. They simply assert that if we examine each of the boxes, we will come upon a box that contains more than one object. The pigeonhole principle merely guarantees the existence of such a box. Thus, whenever the pigeonhole principle is applied to prove the existence of an arrangement or some phenomenon, it will give no indication of how to construct the arrangement or find an instance of the phenomenon other than to examine all possibilities.

[1]The word *shoebox* is a mistranslation and folk etymology for the German *Schubfach*, which means "pigeonhole" (in a desk).

Notice also that the conclusion of the pigeonhole principle cannot be *guaranteed* if there are only n (or fewer) objects. This is because we may put a different object in each of the n boxes. Of course, it is possible to distribute as few as two objects among the boxes in such a way that a box contains two objects, but there is no guarantee that a box will contain two or more objects unless we distribute at least $n+1$ objects. The pigeonhole principle asserts that, no matter how we distribute $n+1$ objects among n boxes, we cannot avoid putting two objects in the same box.

Instead of putting objects into boxes, we may think of coloring each object with one of n colors. The pigeonhole principle asserts that if $n+1$ objects are colored with n colors, then two objects have the same color.

We begin with two simple applications:

Application 1. Among 13 people there are 2 who have their birthdays in the same month. □

Application 2. There are n married couples. How many of the $2n$ people must be selected to guarantee that a married couple has been selected?

To apply the pigeonhole principle in this case, think of n boxes, one corresponding to each of the n couples. If we select $n+1$ people and put each of them in the box corresponding to the couple to which they belong, then some box contains two people; that is, we have selected a married couple. Two of the ways to select n people without getting a married couple are to select all the husbands or all the wives. Therefore, $n+1$ is the smallest number that will guarantee a married couple has been selected. □

There are other principles related to the pigeonhole principle that are worth stating formally:

- *If n objects are put into n boxes and no box is empty, then each box contains exactly one object.*
- *If n objects are put into n boxes and no box gets more than one object, then each box has an object in it.*

Referring to Application 2, if we select n people in such a way that we have selected at least one person from each married couple, then we have selected exactly one person from each couple. Also, if we select n people without selecting more than one person from each married couple, then we have selected at least one (and, hence, exactly one) person from each couple.

More abstract formulations of the three principles enunciated thus far are as follows:

Let X and Y be finite sets and let $f : X \to Y$ be a function from X to Y.

- *If X has more elements than Y, then f is not one-to-one.*
- *If X and Y have the same number of elements and f is onto, then f is one-to-one.*
- *If X and Y have the same number of elements and f is one-to-one, then f is onto.*

Application 3. Given m integers $a_1, a_2, \ldots, a_m$, there exist integers k and l with $0 \leq k < l \leq m$ such that $a_{k+1} + a_{k+2} + \cdots + a_l$ is divisible by m. Less formally, there exist consecutive a's in the sequence $a_1, a_2, \ldots, a_m$ whose sum is divisible by m.

To see this, consider the m sums

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + a_3 + \cdots + a_m.$$

If any of these sums is divisible by m, then the conclusion holds. Thus, we may suppose that each of these sums has a nonzero remainder when divided by m, and so a remainder equal to one of $1, 2, \ldots, m-1$. Since there are m sums and only $m-1$ remainders, two of the sums have the same remainder when divided by m. Therefore, there are integers k and l with $k < l$ such that $a_1 + a_2 + \cdots + a_k$ and $a_1 + a_2 + \cdots + a_l$ have the same remainder r when divided by m:

$$a_1 + a_2 + \cdots + a_k = bm + r, \qquad a_1 + a_2 + \cdots + a_l = cm + r.$$

Subtracting, we find that $a_{k+1} + \cdots + a_l = (c-b)m$; thus, $a_{k+1} + \cdots + a_l$ is divisible by m.

To illustrate this argument,[2] let $m = 7$ and let our integers be $2, 4, 6, 3, 5, 5$, and 6. Computing the sums as before, we get $2, 6, 12, 15, 20$, 25, and 31 whose remainders when divided by 7 are, respectively, $2, 6, 5, 1$, $6, 4$, and 3. We have two remainders equal to 6, and this implies the conclusion that $6 + 3 + 5 = 14$ is divisible by 7. □

Application 4. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, to avoid tiring himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of (consecutive) days during which the chess master will have played *exactly* 21 games.

Let a_1 be the number of games played on the first day, a_2 the total number of games played on the first and second days, a_3 the total number of games played on the first, second, and third days, and so on. The sequence of numbers $a_1, a_2, \ldots, a_{77}$ is a strictly increasing sequence[3] since at least one game is played each day. Moreover, $a_1 \geq 1$,

[2] The argument actually contains a nice algorithm, whose validity relies on the pigeonhole principle, for finding the consecutive a's, which is more efficient than examining all sums of consecutive a's.

[3] Each term of the sequence is larger than the one that precedes it.

and since at most 12 games are played during any one week, $a_{77} \leq 12 \times 11 = 132$.[4] Hence, we have

$$1 \leq a_1 < a_2 < \cdots < a_{77} \leq 132.$$

The sequence $a_1 + 21,\ a_2 + 21, \ldots, a_{77} + 21$ is also a strictly increasing sequence:

$$22 \leq a_1 + 21 < a_2 + 21 < \cdots < a_{77} + 21 \leq 132 + 21 = 153.$$

Thus each of the 154 numbers

$$a_1, a_2, \ldots, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$$

is an integer between 1 and 153. It follows that two of them are equal. Since no two of the numbers $a_1, a_2, \ldots, a_{77}$ are equal and no two of the numbers $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ are equal, there must be an i and a j such that $a_i = a_j + 21$. Therefore, on days $j + 1, j + 2, \ldots, i$ the chess master played a total of 21 games. □

Application 5. From the integers $1, 2, \ldots, 200$, we choose 101 integers. Show that, among the integers chosen, there are two such that one of them is divisible by the other.

By factoring out as many 2s as possible, we see that any integer can be written in the form $2^k \times a$, where $k \geq 0$ and a is odd. For an integer between 1 and 200, a is one of the 100 numbers $1, 3, 5, \ldots, 199$. Thus among the 101 integers chosen, there are two having a's of equal value when written in this form. Let these two numbers be $2^r \times a$ and $2^s \times a$. If $r < s$, then the second number is divisible by the first. If $r > s$, then the first is divisible by the second. □

Let us note that the result of Application 5 is the best possible in the sense that we may select 100 integers from $1, 2, \ldots, 200$ in such a way that no one of the selected integers is divisible by any other (for instance, the 100 integers $101, 102, \ldots, 199, 200$).

We conclude this section with another application from number theory. First, we recall that two positive integers m and n are said to be *relatively prime* if their greatest common divisor[5] is 1. Thus 12 and 35 are relatively prime, but 12 and 15 are not since 3 is a common divisor of 12 and 15.

Application 6. (*Chinese remainder theorem*) Let m and n be relatively prime positive integers, and let a and b be integers where $0 \leq a \leq m - 1$ and $0 \leq b \leq n - 1$. Then there is a positive integer x such that the remainder when x is divided by m is a, and the remainder when x is divided by n is b; that is, x can be written in the form $x = pm + a$ and also in the form $x = qn + b$ for some integers p and q.

[4]This is the only place where the assumption that at most 12 games are played during any of the 11 calendar weeks is used. Thus, this assumption could be replaced by the assumption that at most 132 games are played in 77 days.

[5]Also called *greatest common factor* or *highest common factor.*

To show this, we consider the n integers

$$a, m+a, 2m+a, \ldots, (n-1)m+a.$$

Each of these integers has remainder a when divided by m. Suppose that two of them had the same remainder r when divided by n. Let the two numbers be $im+a$ and $jm+a$, where $0 \leq i < j \leq n-1$. Then there are integers q_i and q_j such that

$$im + a = q_i n + r$$

and

$$jm + a = q_j n + r.$$

Subtracting the first equation from the second, we get

$$(j-i)m = (q_j - q_i)n.$$

The preceding equation tells us that n is a factor of the number $(j-i)m$. Since n has no common factor other than 1 with m, it follows that n is a factor of $j-i$. However, $0 \leq i < j \leq n-1$ implies that $0 < j-i \leq n-1$, and hence n cannot be a factor of $j-i$. This contradiction arises from our supposition that two of the numbers

$$a, m+a, 2m+a, \ldots, (n-1)m+a$$

had the same remainder when divided by n. We conclude that each of these n numbers has a different remainder when divided by n. By the pigeonhole principle, each of the n numbers $0, 1, \ldots, n-1$ occurs as a remainder; in particular, the number b does. Let p be the integer with $0 \leq p \leq n-1$ such that the number $x = pm + a$ has remainder b when divided by n. Then, for some integer q,

$$x = qn + b.$$

So $x = pm + a$ and $x = qn + b$, and x has the required properties. □

The fact that a rational number a/b has a decimal expansion that eventually repeats is a consequence of the pigeonhole principle, and we leave a proof of this fact for the Exercises.

For further applications we need a stronger form of the pigeonhole principle.

3.2 Pigeonhole Principle: Strong Form

The following theorem contains Theorem 3.1.1 as a special case:

Theorem 3.2.1 *Let $q_1, q_2, \ldots, q_n$ be positive integers. If*

$$q_1 + q_2 + \cdots + q_n - n + 1$$

objects are distributed into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, ... , or the nth box contains at least q_n objects.

Proof. Suppose that we distribute $q_1 + q_2 + \cdots + q_n - n + 1$ objects among n boxes. If for each $i = 1, 2, \ldots, n$ the ith box contains fewer than q_i objects, then the total number of objects in all boxes does not exceed

$$(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1) = q_1 + q_2 + \cdots + q_n - n.$$

Since this number is one less than the number of objects distributed, we conclude that for some $i = 1, 2, \ldots, n$ the ith box contains at least q_i objects. □

Notice that it is possible to distribute $q_1 + q_2 + \cdots + q_n - n$ objects among n boxes in such a way that for no $i = 1, 2, \ldots, n$ is it true that the ith box contains q_i or more objects. We do this by putting $q_1 - 1$ objects into the first box, $q_2 - 1$ objects into the second box, and so on.

The simple form of the pigeonhole principle is obtained from the strong form by taking $q_1 = q_2 = \cdots = q_n = 2$. Then

$$q_1 + q_2 + \cdots + q_n - n + 1 = 2n - n + 1 = n + 1.$$

In terms of coloring, the strong form of the pigeonhole principle asserts that if each of $q_1 + q_2 + \cdots + q_n - n + 1$ objects is assigned one of n colors, then there is an i such that there are (at least) q_i objects of the ith color.

In elementary mathematics the strong form of the pigeonhole principle is most often applied in the special case when $q_1, q_2, \ldots, q_n$ are all equal to some integer r. We formulate this special case as a corollary.

Corollary 3.2.2 *Let n and r be positive integers. If $n(r-1)+1$ objects are distributed into n boxes, then at least one of the boxes contains r or more of the objects.*

Another way to formulate the assertion in this corollary is as an averaging principle:

> If the average of n nonnegative integers $m_1, m_2, \ldots, m_n$ is greater than $r - 1$, that is,
>
> $$\frac{m_1 + m_2 + \cdots + m_n}{n} > r - 1,$$
>
> then at least one of the integers is greater than or equal to r.

The connection between the assertion in Corollary 3.2.2 and this averaging principle is seen by taking $n(r-1)+1$ objects and putting them into n boxes. For $i = 1, 2, \ldots, n$, let m_i be the number of objects in the ith box. Then the average of the numbers $m_1, m_2, \ldots, m_n$ is

$$\frac{m_1 + m_2 + \cdots + m_n}{n} = \frac{n(r-1)+1}{n} = (r-1) + \frac{1}{n}.$$

Since this average is greater than $r-1$, one of the integers m_i is at least r. In other words, one of the boxes contains at least r objects.

A different averaging principle is the following:

If the average of n nonnegative integers $m_1, m_2, \ldots, m_n$ is less than $r+1$, that is,

$$\frac{m_1 + m_2 + \cdots + m_n}{n} < r + 1,$$

then at least one of the integers is less than $r+1$.

Application 7. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket to guarantee that either there are at least eight apples or at least six bananas or at least nine oranges?

By the strong form of the pigeonhole principle, $8+6+9-3+1 = 21$ pieces of fruit, no matter how selected, will guarantee a basket of fruit with the desired properties. But 7 apples, 5 bananas, and 8 oranges, a total of 20 pieces of fruit, will not. □

The following is yet another averaging principle:

- If the average of n nonnegative integers $m_1, m_2, \ldots, m_n$ is at least equal to r, then at least one of the integers $m_1, m_2, \ldots, m_n$ satisfies $m_i \geq r$.

Application 8. Two disks, one smaller than the other, are each divided into 200 congruent sectors.[6] In the larger disk, 100 of the sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk, each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The small disk is then placed on the larger disk so that their centers coincide. Show that it is possible to align the two disks so that the number of sectors of the small disk whose color matches the corresponding sector of the large disk is at least 100.

To see this, we observe that if the large disk is fixed in place, there are 200 possible positions for the small disk such that each sector of the small disk is contained in a sector of the large disk. We first count the total number of color matches over all of

[6] Two hundred equal slices of a pie.

the 200 possible positions of the disks. Since the large disk has 100 sectors of each of the two colors, each sector of the small disk will match in color the corresponding sector of the large disk in exactly 100 of the 200 possible positions. Thus, the total number of color matches over all the positions equals the number of sectors of the small disk multiplied by 100, and this equals 20,000. Therefore, the average number of color matches per position is 20,000/200=100. So there must be some position with at least 100 color matches. □

We next present an application that was first discovered by Erdös and Szekeres.[7]

Application 9. Show that every sequence $a_1, a_2, \ldots, a_{n^2+1}$ of n^2+1 real numbers contains either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

We first clarify the notion of a subsequence. If $b_1, b_2, \ldots, b_m$ is a sequence, then $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$ is a *subsequence*, provided that $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. Thus b_2, b_4, b_5, b_6 is a subsequence of $b_1, b_2, \ldots, b_8$, but b_2, b_6, b_5 is not. The subsequence $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$ is *increasing* (more properly *not decreasing*) if $b_{i_1} \leq b_{i_2} \leq \cdots \leq b_{i_k}$ and *decreasing* if $b_{i_1} \geq b_{i_2} \geq \cdots \geq b_{i_k}$.

We now prove the assertion. We suppose that there is no increasing subsequence of length $n+1$ and show that there must be a decreasing subsequence of length $n+1$. For each $k = 1, 2, \ldots, n^2+1$, let m_k be the length of the longest increasing subsequence that begins with a_k. Suppose $m_k \leq n$ for each $k = 1, 2, \ldots, n^2+1$, so that there is no increasing subsequence of length $n+1$. Since $m_k \geq 1$ for each $k = 1, 2, \ldots, n^2+1$, the numbers $m_1, m_2, \ldots, m_{n^2+1}$ are n^2+1 integers each between 1 and n. By the strong form of the pigeonhole principle, $n+1$ of the numbers $m_1, m_2, \ldots, m_{n^2+1}$ are equal. Let

$$m_{k_1} = m_{k_2} = \cdots = m_{k_{n+1}},$$

where $1 \leq k_1 < k_2 < \cdots < k_{n+1} \leq n^2+1$. Suppose that for some $i = 1, 2, \ldots, n$, $a_{k_i} < a_{k_{i+1}}$. Then, since $k_i < k_{i+1}$ we could take a longest increasing subsequence beginning with $a_{k_{i+1}}$ and put a_{k_i} in front to obtain an increasing subsequence beginning with a_{k_i}. Since this implies that $m_{k_i} > m_{k_{i+1}}$, we conclude that $a_{k_i} \geq a_{k_{i+1}}$. Since this is true for each $i = 1, 2, \ldots, n$, we have

$$a_{k_1} \geq a_{k_2} \geq \cdots \geq a_{k_{n+1}},$$

and we conclude that $a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}}$ is a decreasing subsequence of length $n+1$. □

An amusing formulation of Application 9 is the following: Suppose that n^2+1 people are lined up shoulder to shoulder in a straight line. Then it is always possible to choose $n+1$ of the people to take one step forward so that, going from left to right,

[7] P. Erdös and A. Szekeres, A Combinatorial Problem in Geometry, *Compositio Mathematica*, 2 (1935), 463–470.

either their heights are increasing or their heights are decreasing. It is instructive to read through the proof of Application 9 in these terms.

3.3 A Theorem of Ramsey

We now discuss a profound and important generalization of the pigeonhole principle called Ramsey's theorem, after the English logician Frank Ramsey.[8]

The following is the most popular and easily understood instance of Ramsey's theorem:

> Of six (or more) people, either there are three, each pair of whom are acquainted, or there are three, each pair of whom are unacquainted.

One way to prove this result is to examine all the different ways in which six people can be acquainted and unacquainted. This is a tedious task, but nonetheless one that can be accomplished with a little fortitude. There is, however, a simple and elegant proof that avoids consideration of cases. Before giving this proof, we formulate the result more abstractly as

$$K_6 \rightarrow K_3, K_3 \qquad \text{(read } K_6 \textit{ arrows } K_3, K_3\text{)}. \tag{3.1}$$

What does this mean? First, by K_6 we mean a set of six objects (e.g., people) and all of the 15 (unordered) pairs of these objects. We can picture K_6 by choosing six points in the plane, no three of which are collinear, and then drawing the edge or line segment connecting each pair of points (the edges now represent the pairs). In general, we mean by K_n a set of n objects and all of the pairs of these objects.[9] Illustrations for K_n ($n = 1, 2, 3, 4, 5$) are given in Figure 3.1. Notice that the picture of K_3 is that of a triangle, and we often refer to K_3 as a *triangle*.

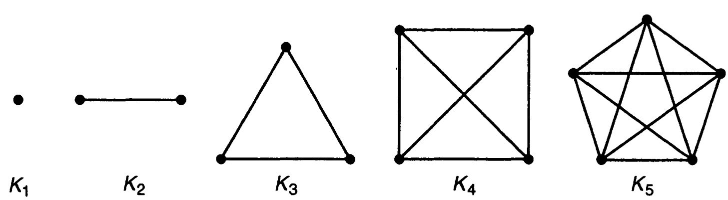

Figure 3.1

We distinguish between acquainted pairs and unacquainted pairs by coloring edges red for acquainted and blue for unacquainted. "Three mutually acquainted people"

[8]Frank Ramsey was born in 1903 and died in 1930 when he was not quite 27 years of age. In spite of his premature death, he laid the foundation for what is now called *Ramsey theory*.

[9]In later chapters, K_n is called the *complete graph* of order n.

now means "a K_3 each of whose edges is colored red: a *red* K_3." Similarly, three mutually unacquainted people form a *blue* K_3. We can now explain the expression (3.1):

> $K_6 \to K_3, K_3$ is the assertion that *no matter how the edges of* K_6 *are colored with the colors red and blue, there is always a red* K_3 *(three of the original six points with the three line segments between them all colored red) or a blue* K_3 *(three of the original six points with the three line segments between them all colored blue), in short, a monochromatic triangle.*

To prove that $K_6 \to K_3, K_3$, we argue as follows: Suppose the edges of K_6 have been colored red or blue in any way. Consider one of the points p of K_6. It meets five edges. Since each of these five edges is colored red or blue, it follows (from the strong form of the pigeonhole principle) that either at least three of them are colored red or at least three of them are colored blue. We suppose that three of the five edges meeting the point p are red. (If three are blue, a similar argument works.) Let the three red edges meeting p join p to points $a, b,$ and c, respectively. Consider the edges which join a, b, c in pairs. If all of these are blue, then a, b, c determine a blue K_3. If one of them, say the one joining a and b, is red, then p, a, b determine a red K_3. Thus, we are guaranteed either a red K_3 or a blue K_3.

We observe that the assertion $K_5 \to K_3, K_3$ is false. This is because there is *some* way to color the edges of K_5 without creating a red K_3 or a blue K_3. This is shown in Figure 3.2, where the edges of the pentagon (the solid edges) are the red edges and the edges of the inscribed pentagram (the dashed edges) are the blue edges.

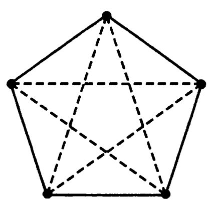

Figure 3.2

We now state and prove Ramsey's theorem, although still not in its full generality.

Theorem 3.3.1 *If* $m \geq 2$ *and* $n \geq 2$ *are integers, then there is a positive integer* p *such that*

$$K_p \to K_m, K_n.$$

In words, Ramsey's theorem asserts that given m and n there is a positive integer p such that, if the edges of K_p are colored red or blue, then either there is a red K_m

or there is a blue K_n. The existence of either a red K_m or a blue K_n is guaranteed, no matter how the edges of K_p are colored. If $K_p \to K_m, K_n$, then $K_q \to K_m, K_n$ for every integer $q \geq p$. The *Ramsey number* $r(m,n)$ is the smallest integer p such that $K_p \to K_m, K_n$. Thus *Ramsey's theorem asserts the existence of the number* $r(m,n)$. By interchanging the colors red and blue, we see that

$$r(m,n) = r(n,m).$$

The facts that $K_6 \to K_3, K_3$ and $K_5 \not\to K_3, K_3$ imply that

$$r(3,3) = 6.$$

The Ramsey numbers $r(2,n)$ and $r(m,2)$ are easy to determine. We show that $r(2,n) = n$:

$r(2,n) \leq n$: If we color the edges of K_n either red or blue, then either some edge is colored red (and so we have a red K_2) or all edges are blue (and so we have a blue K_n).

$r(2,n) > n-1$: If we color all the edges of K_{n-1} blue, then we have neither a red K_2 nor a blue K_n.

In a similar way, we show that $r(m,2) = m$. The numbers $r(2,n)$ and $r(m,2)$ with $m,n \geq 2$ are the *trivial Ramsey numbers.*

Proof of Theorem 3.3.1. We show the existence of the numbers $r(m,n)$ by using (double) induction on both integer parameters $m \geq 2$ and $n \geq 2$. If $m = 2$, we know that $r(2,n) = n$, and if $n = 2$, we know that $r(m,2) = m$. We now assume that $m \geq 3$ and $n \geq 3$, and take as our inductive assumption that both $r(m-1,n)$ and $r(m,n-1)$ exist. Let $p = r(m-1,n) + r(m,n-1)$. We will show that $K_p \to K_m, K_n$ for this integer p.

Suppose that the edges of K_p have been colored red or blue in any way. Consider one of the points x of K_n. Let R_x be the set of points that are joined to x by a red edge, and let B_x be the set of points that are joined to x by a blue edge. Then

$$|R_x| + |B_x| = p - 1 = r(m-1,n) + r(m,n-1) - 1,$$

implying that

(1) $|R_x| \geq r(m-1,n)$, or

(2) $|B_x| \geq r(m,n-1)$.

(If both (1) and (2) failed, then $|R_x| + |B_x| \leq r(m-1,n) - 1 + r(m,n-1) - 1 = p-2$, a contradiction.)

Suppose that (1) holds. Let $q = |R_x|$ so that $q \geq r(m-1,n)$. Then considering K_q on the points of R_x, we see that either there are $m-1$ points of K_q (and so of

K_p) all of whose edges are colored red (that is, a red K_{m-1}) or there are n points all of whose edges are colored blue (that is, a blue K_n). If the second possibility holds, we are done since we have a blue K_n. If the first possibility holds, we are also done since we can take the red K_{m-1} and add the point x to it to obtain a red K_m, since all edges joining x to the points in R_x are colored red.

A similar argument works when (2) holds. We conclude by induction that the numbers $r(m,n)$ exist for all integers $m, n \geq 2$. □

Our proof of Theorem 3.3.1 not only shows that the Ramsey numbers $r(m,n)$ exist, but also that they satisfy the inequality

$$r(m,n) \leq r(m-1,n) + r(m,n-1) \quad (m,n \geq 3). \tag{3.2}$$

Let

$$f(m,n) = \binom{m+n-2}{m-1} \quad (m,n \geq 2).$$

Then, using Pascal's formula, we get that

$$\binom{m+n-2}{m-1} = \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2}.$$

Hence

$$f(m,n) = f(m-1,n) + f(m,n-1) \quad (m,n \geq 3),$$

a relation similar to that of (3.2) but with equality: Since $r(2,n) = n = f(2,n)$ and $r(m,2) = m = f(m,2)$, we conclude that the Ramsey number $r(m,n)$ satisfies

$$r(m,n) \leq \binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}.$$

The following list[10] contains known facts about nontrivial Ramsey numbers $r(m,n)$:

[10]The paper "Small Ramsey Numbers" by S.P. Radziszowski, *Electronic Journal of Combinatorics*, Dynamic Survey #1, contains this and other information; see http://www.combinatorics.org.

$$\begin{aligned}
&r(3,3) = 6,\\
&r(3,4) = r(4,3) = 9,\\
&r(3,5) = r(5,3) = 14,\\
&r(3,6) = r(6,3) = 18,\\
&r(3,7) = r(7,3) = 23,\\
&r(3,8) = r(8,3) = 28,\\
&r(3,9) = r(9,3) = 36,\\
&40 \leq r(3,10) = r(10,3) \leq 43,\\
&r(4,4) = 18,\\
&r(4,5) = r(5,4) = 25,\\
&35 \leq r(4,6) = r(6,4) \leq 41\\
&43 \leq r(5,5) \leq 49\\
&58 \leq r(5,6) = r(6,5) \leq 87\\
&102 \leq r(6,6) \leq 165.
\end{aligned}$$

Notice that the fact that $r(3,10)$ lies between 40 and 43 implies that

$$K_{43} \to K_3, K_{10}$$

and

$$K_{39} \not\to K_3, K_{10}.$$

Thus, there is no way to color the edges of K_{43} without creating either a red K_3 or a blue K_{10}; there is a way to color the edges of K_{39} without creating either a red K_3 or a blue K_{10}, but neither of these conclusions is known to be true for K_{40}, K_{41}, and K_{42}. The assertion $43 \leq r(5,5) \leq 49$ implies that $K_{59} \to K_5, K_5$ and that there is a way to color the edges of K_{42} without creating a monochromatic K_5.

Ramsey's theorem generalizes to any number of colors. We give a very brief introduction. If n_1, n_2, and n_3 are integers greater than or equal to 2, then there exists an integer p such that

$$K_p \to K_{n_1}, K_{n_2}, K_{n_3}.$$

In words, if each of the edges of K_p is colored red, blue, or green, then either there is a red K_{n_1} or a blue K_{n_2} or a green K_{n_3}. The smallest integer p for which this assertion holds is the Ramsey number $r(n_1, n_2, n_3)$. The only nontrivial Ramsey number of this type that is known is

$$r(3,3,3) = 17.$$

Thus $K_{17} \to K_3, K_3, K_3$ but $K_{16} \not\to K_3, K_3, K_3$. The Ramsey numbers $r(n_1, n_2, \ldots, n_k)$ are defined in a similar way, and Ramsey's theorem in its full generality for pairs asserts that these numbers exist; that is, there is an integer p such that

$$K_p \to K_{n_1}, K_{n_2}, \ldots, K_{n_k}.$$

There is an even more general form of Ramsey's theorem in which pairs (subsets of two elements) are replaced by subsets of t elements for some fixed integer $t \geq 1$. Let

$$K_n^t$$

denote the collection of all subsets of t elements of a set of n elements. Generalizing our preceding notation, we obtain the general form of Ramsey's theorem:

Given integers $t \geq 2$ and integers $q_1, q_2, \ldots, q_k \geq t$, there exists an integer p such that

$$K_p^t \to K_{q_1}^t, K_{q_2}^t, \ldots, K_{q_k}^t.$$

In words, there exists an integer p such that if each of the t-element subsets of a p-element set is assigned one of k colors $c_1, c_2, \ldots, c_k$, then either there are q_1 elements, all of whose t-element subsets are assigned the color c_1, or there are q_2 elements, all of whose t-element subsets are assigned the color c_2, ..., or there are q_k elements, all of whose t-element subsets are assigned the color c_k. The smallest such integer p is the Ramsey number

$$r_t(q_1, q_2, \ldots, q_k).$$

Suppose $t = 1$. Then $r_1(q_1, q_2, \ldots, q_k)$ is the smallest number p such that, if the elements of a set of p elements are colored with one of the colors $c_1, c_2, \ldots, c_k$, then either there are q_1 elements of color c_1, or q_2 elements of color c_2, . . . , or q_k elements of color c_k. Thus, by the strong form of the pigeonhole principle,

$$r_1(q_1, q_2, \ldots, q_k) = q_1 + q_2 + \cdots q_k - k + 1.$$

This demonstrates that Ramsey's theorem is a generalization of the strong form of the pigeonhole principle.

The determination of the general Ramsey numbers $r_t(q_1, q_2, \ldots, q_k)$ is a difficult problem. Very little is known about their exact values. It is not difficult to see that

$$r_t(t, q_2, \ldots, q_k) = r_t(q_2, \ldots, q_k)$$

and that the order in which $q_1, q_2, \ldots, q_k$ are listed does not affect the value of the Ramsey number.

3.4 Exercises

1. Concerning Application 4, show that there is a succession of days during which the chess master will have played exactly k games, for each $k = 1, 2, \ldots, 21$. (The case $k = 21$ is the case treated in Application 4.) Is it possible to conclude that there is a succession of days during which the chess master will have played exactly 22 games?

2. $*$ Concerning Application 5, show that if 100 integers are chosen from $1, 2, \ldots, 200$, and one of the integers chosen is less than 16, then there are two chosen numbers such that one of them is divisible by the other.

3. Generalize Application 5 by choosing (how many?) integers from the set

$$\{1, 2, \ldots, 2n\}.$$

4. Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, then there are always two which differ by 1.

5. Show that if $n+1$ distinct integers are chosen from the set $\{1, 2, \ldots, 3n\}$, then there are always two which differ by at most 2.

6. Generalize Exercises 4 and 5.

7. $*$ Show that for any given 52 integers there exist two of them whose sum, or else whose difference, is divisible by 100.

8. Use the pigeonhole principle to prove that the decimal expansion of a rational number m/n eventually is repeating. For example,

$$\frac{34,478}{99,900} = 0.345125125125125 12 \cdots.$$

9. In a room there are 10 people, none of whom are older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that we can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?

10. A child watches TV at least one hour each day for seven weeks but, because of parental rules, never more than 11 hours in any one week. Prove that there is some period of consecutive days in which the child watches exactly 20 hours of TV. (It is assumed that the child watches TV for a whole number of hours each day.)

11. A student has 37 days to prepare for an examination. From past experience she knows that she will require no more than 60 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day, however), there is a succession of days during which she will have studied exactly 13 hours.

12. Show by example that the conclusion of the Chinese remainder theorem (Application 6) need not hold when m and n are not relatively prime.

13. * Let S be a set of six points in the plane, with no three of the points collinear. Color either red or blue each of the 15 line segments determined by the points of S. Show that there are at least two triangles determined by points of S which are either red triangles or blue triangles. (Both may be red, or both may be blue, or one may be red and the other blue.)

14. A bag contains 100 apples, 100 bananas, 100 oranges, and 100 pears. If I pick one piece of fruit out of the bag every minute, how long will it be before I am assured of having picked at least a dozen pieces of fruit of the same kind?

15. Prove that, for any $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$, there exist two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

16. Prove that in a group of $n > 1$ people there are two who have the same number of acquaintances in the group. (It is assumed that no one is acquainted with oneself.)

17. There are 100 people at a party. Each person has an even number (possibly zero) of acquaintances. Prove that there are three people at the party with the same number of acquaintances.

18. Prove that of any five points chosen within a square of side length 2, there are two whose distance apart is at most $\sqrt{2}$.

19. (a) Prove that of any five points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{2}$.
    (b) Prove that of any 10 points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{3}$.
    (c) Determine an integer m_n such that if m_n points are chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $1/n$.

20. Prove that $r(3,3,3) \leq 17$.

21. * Prove that $r(3,3,3) \geq 17$ by exhibiting a coloring, with colors red, blue, and green, of the line segments joining 16 points with the property that there do not exist three points such that the three line segments joining them are all colored the same.

22. Prove that

$$r(\underbrace{3,3,\ldots,3}_{k+1}) \leq (k+1)(r(\underbrace{3,3,\ldots,3}_{k}) - 1) + 2.$$

Use this result to obtain an upper bound for

$$r(\underbrace{3,3,\ldots,3}_{n}).$$

23. The line segments joining 10 points are arbitrarily colored red or blue. Prove that there must exist three points such that the three line segments joining them are all red, or four points such that the six line segments joining them are all blue (that is, $r(3,4) \leq 10$).

24. Let q_3 and t be positive integers with $q_3 \geq t$. Determine the Ramsey number $r_t(t,t,q_3)$.

25. Let $q_1, q_2, \ldots, q_k, t$ be positive integers, where $q_1 \geq t$, $q_2 \geq t, \ldots,$ $q_k \geq t$. Let m be the largest of $q_1, q_2, \ldots, q_k$. Show that

$$r_t(m, m, \ldots, m) \geq r_t(q_1, q_2, \ldots, q_k).$$

Conclude that, to prove Ramsey's theorem, it is enough to prove it in the case that $q_1 = q_2 = \cdots = q_k$.

26. Suppose that the mn people of a marching band are standing in a rectangular formation of m rows and n columns in such a way that in each row each person is taller than the one to his or her left. Suppose that the leader rearranges the people in each column in increasing order of height from front to back. Show that the rows are still arranged in increasing order of height from left to right.

27. A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

28. At a dance party there are 100 men and 20 women. For each i from $1, 2, \ldots, 100$, the ith man selects a group of a_i women as potential dance partners (his "dance list," if you will), but in such a way that given any group of 20 men, it is always possible to pair the 20 men with the 20 women, with each man paired with a woman on his dance list. What is the smallest sum $a_1 + a_2 + \cdots + a_{100}$ for which there is a selection of dance lists that will *guarantee* this?

29. A number of different objects have been distributed into n boxes $B_1, B_2, \ldots, B_n$. All the objects from these boxes are removed and redistributed into $n+1$ new boxes $B_1^*, B_2^*, \ldots, B_{n+1}^*$, with no new box empty (so the total number of objects must be at least $n+1$). Prove that there are two objects each of which has the property that it is in a new box that contains fewer objects than the old box that contained it.