## 1.2 Sets and Multisets

We have (finally!) completed our description of the solution of an enumerative problem, and we are now ready to delve into some actual problems. Let us begin with the basic problem of counting subsets of a set. Let $S = \{x_1, x_2, \ldots, x_n\}$ be an $n$-element set, or *$n$-set* for short. Let $2^S$ denote the set of all subsets of $S$, and let $\{0,1\}^n = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) : \varepsilon_i = 0 \text{ or } 1\}$. Since there are two possible values for each $\varepsilon_i$, we have $\#\{0,1\}^n = 2^n$. Define a map $\theta : 2^S \to \{0,1\}^n$ by $\theta(T) = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, where

$$\varepsilon_i = \begin{cases} 1, & \text{if } x_i \in T \\ 0, & \text{if } x_i \notin T. \end{cases}$$

For example, if $n = 5$ and $T = \{x_2, x_4, x_5\}$, then $\theta(T) = (0,1,0,1,1)$. Most readers will realize that $\theta(T)$ is just the *characteristic vector* of $T$. It is easily seen that $\theta$ is a bijection, so that we have given a combinatorial proof that $\#2^S = 2^n$. Of course there are many alternative proofs of this simple result, and many of these proofs could be regarded as combinatorial.

Now define $\binom{S}{k}$ (sometimes denoted $S^{(k)}$ or otherwise, and read "$S$ choose $k$") to be the set of all $k$-element subsets (or *$k$-subsets*) of $S$, and *define* $\binom{n}{k} = \#\binom{S}{k}$, read "$n$ choose $k$" (ignore our previous use of the symbol $\binom{n}{k}$) and called a *binomial coefficient.* Our goal is to prove the formula

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{1.16}$$

Note that if $0 \le k \le n$ then the right-hand side of equation (1.16) can be rewritten $n!/k!(n-k)!$. The right-hand side of (1.16) can be used to define $\binom{n}{k}$ for any complex number (or indeterminate) $n$, provided $k \in \mathbb{N}$. The numerator $n(n-1)\cdots(n-k+1)$ of (1.16) is read "$n$ lower factorial $k$" and is denoted $(n)_k$. CAVEAT. Many mathematicians, especially those in the theory of special functions, use the notation $(n)_k = n(n+1)\cdots(n+k-1)$.

We would like to give a bijective proof of (1.16), but the factor $k!$ in the denominator makes it difficult to give a "simple" interpretation of the right-hand side. Therefore we use the standard technique of clearing the denominator. To this end we count in two ways the number $N(n,k)$ of ways of choosing a $k$-subset $T$ of $S$ and then linearly ordering the elements of $T$. We can pick $T$ in $\binom{n}{k}$ ways, then pick an element of $T$ in $k$ ways to be first in the ordering, then pick another element in $k-1$ ways to be second, and so on. Thus

$$N(n,k) = \binom{n}{k} k!.$$

On the other hand, we could pick any element of $S$ in $n$ ways to be first in the ordering, then another element in $n-1$ ways to be second, on so on, down to any remaining element in $n-k+1$ ways to be $k$th. Thus

$$N(n,k) = n(n-1)\cdots(n-k+1).$$

We have therefore given a combinatorial proof that

$$\binom{n}{k}k! = n(n-1)\cdots(n-k+1),$$

and hence of equation (1.16).

A generating function approach to binomial coefficients can be given as follows. Regard $x_1, \ldots, x_n$ as independent indeterminates. It is an immediate consequence of the process of multiplication (one could also give a rigorous proof by induction) that

$$(1+x_1)(1+x_2)\cdots(1+x_n) = \sum_{T\subseteq S}\prod_{x_i\in T} x_i. \tag{1.17}$$

If we put each $x_i = x$, then we obtain

$$(1+x)^n = \sum_{T\subseteq S}\prod_{x_i\in T} x = \sum_{T\subseteq S} x^{\#T} = \sum_{k\geq 0}\binom{n}{k}x^k, \tag{1.18}$$

since the term $x^k$ appears exactly $\binom{n}{k}$ times in the sum $\sum_{T\subseteq S} x^{\#T}$. This reasoning is an instance of the simple but useful observation that if $\mathcal{S}$ is a collection of finite sets such that $\mathcal{S}$ contains exactly $f(n)$ sets with $n$ elements, then

$$\sum_{S\in\mathcal{S}} x^{\#S} = \sum_{n\geq 0} f(n)x^n.$$

Somewhat more generally, if $g : \mathbb{N} \to \mathbb{C}$ is any function, then

$$\sum_{S\in\mathcal{S}} g(\#S)x^{\#S} = \sum_{n\geq 0} g(n)f(n)x^n.$$

Equation (1.18) is such a simple result (the binomial theorem for the exponent $n \in \mathbb{N}$) that it is hardly necessary to obtain first the more refined (1.17). However, it is often easier in dealing with generating functions to work with the most number of variables (indeterminates) possible and then specialize. Often the more refined formula will be more transparent, and its various specializations will be automatically unified.

Various identities involving binomial coefficients follow easily from the identity $(1+x)^n = \sum_{k\geq 0}\binom{n}{k}x^k$, and the reader will find it instructive to find combinatorial proofs of them. (See Exercise 1.3 for further examples of binomial coefficient identities.) For instance, put $x = 1$ to obtain $2^n = \sum_{k\geq 0}\binom{n}{k}$; put $x = -1$ to obtain $0 = \sum_{k\geq 0}(-1)^k\binom{n}{k}$ if $n > 0$; differentiate and put $x = 1$ to obtain $n2^{n-1} = \sum_{k\geq 0} k\binom{n}{k}$, and so on.

There is a close connection between subsets of a set and compositions of a nonnegative integer. A *composition* of $n$ can be thought of as an expression of $n$ as an *ordered* sum of integers. More precisely, a composition of $n$ is a sequence $\alpha = (a_1, \ldots, a_k)$ of positive integers satisfying $\sum a_i = n$. For instance, there are eight compositions of 4; namely,

$$\begin{array}{ll} 1+1+1+1 & 3+1 \\ 2+1+1 & 1+3 \\ 1+2+1 & 2+2 \\ 1+1+2 & 4. \end{array}$$

If exactly $k$ summands appear in a composition $\alpha$, then we say that $\alpha$ has $k$ *parts*, and we call $\alpha$ a *$k$-composition*. If $\alpha = (a_1, a_2, \ldots, a_k)$ is a $k$-composition of $n$, then define a $(k-1)$-subset $S_\alpha$ of $[n-1]$ by

$$S_\alpha = \{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{k-1}\}.$$

The correspondence $\alpha \mapsto S_\alpha$ gives a bijection between all $k$-compositions of $n$ and $(k-1)$-subsets of $[n-1]$. Hence there are $\binom{n-1}{k-1}$ $k$-compositions of $n$ and $2^{n-1}$ compositions of $n > 0$. The inverse bijection $S_\alpha \mapsto \alpha$ is often represented schematically by drawing $n$ dots in a row and drawing vertical bars between $k-1$ of the $n-1$ spaces separating the dots. This procedure divides the dots into $k$ linearly ordered (from left-to-right) "compartments" whose number of elements is a $k$-composition of $n$. For instance, the compartments

$$\cdot | \cdot\ \cdot | \cdot | \cdot | \cdot\ \cdot\ \cdot | \cdot\ \cdot \tag{1.19}$$

correspond to the 6-composition $(1, 2, 1, 1, 3, 2)$ of 10. The diagram (1.19) illustrates another very general principle related to bijective proofs — it is often efficacious to represent the objects being counted geometrically.

A problem closely related to compositions is that of counting the number $N(n, k)$ of solutions to $x_1 + x_2 + \cdots + x_k = n$ in *nonnegative* integers. Such a solution is called a *weak composition* of $n$ into $k$ parts, or a *weak $k$-composition* of $n$. (A solution in *positive* integers is simply a $k$-composition of $n$.) If we put $y_i = x_i + 1$, then $N(n, k)$ is the number of solutions in positive integers to $y_1 + y_2 + \cdots + y_k = n + k$, that is, the number of $k$-compositions of $n + k$. Hence $N(n, k) = \binom{n+k-1}{k-1}$. A further variant is the enumeration of $\mathbb{N}$-solutions (that is, solutions where each variable lies in $\mathbb{N}$) to $x_1 + x_2 + \cdots + x_k \le n$. Again we use a standard technique, viz., introducing a *slack variable* $y$ to convert the inequality $x_1 + x_2 + \cdots + x_k \le n$ to the equality $x_1 + x_2 + \cdots + x_k + y = n$. An $\mathbb{N}$-solution to this equation is a weak $(k+1)$-composition of $n$, so the number $N(n, k+1)$ of such solutions is $\binom{n+(k+1)-1}{k} = \binom{n+k}{k}$.

A $k$-subset $T$ of an $n$-set $S$ is sometimes called a *$k$-combination of $S$ without repetitions*. This suggests the problem of counting the number of $k$-combinations of $S$ *with repetitions*; that is, we choose $k$ elements of $S$, disregarding order and allowing repeated elements. Denote this number by $\left(\!\binom{n}{k}\!\right)$, which could be read "$n$ multichoose $k$." For instance, if $S = \{1, 2, 3\}$ then the combinations counted by $\left(\!\binom{3}{2}\!\right)$ are 11, 22, 33, 12, 13, 23. Hence $\left(\!\binom{3}{2}\!\right) = 6$. An equivalent but more precise treatment of combinations with repetitions can be made by introducing the concept of a *multiset*. Intuitively, a multiset is a set with repeated elements; for instance, $\{1, 1, 2, 5, 5, 5\}$. More precisely, a *finite multiset* $M$ on a set $S$ is a pair $(S, \nu)$, where $\nu$ is a function $\nu : S \to \mathbb{N}$ such that $\sum_{x \in S} \nu(x) < \infty$. One regards $\nu(x)$ as the number of repetitions of $x$. The integer $\sum_{x \in S} \nu(x)$ is called the *cardinality*, *size*, or *number of elements* of $M$ and is denoted $|M|$, $\#M$, or $\operatorname{card} M$. If $S = \{x_1, \ldots, x_n\}$ and $\nu(x_i) = a_i$, then we call $a_i$ the *multiplicity* of $x_i$ in $M$ and write $M = \{x_1^{a_1}, \ldots, x_n^{a_n}\}$. If $\#M = k$ then we call $M$ a *$k$-multiset*. The set of all $k$-multisets on $S$ is denoted $\left(\!\binom{S}{k}\!\right)$. If $M' = (S, \nu')$ is another multiset on $S$, then we say that $M'$ is a *submultiset* of $M$ if $\nu'(x) \le \nu(x)$ for all $x \in S$. The number of submultisets of $M$ is $\prod_{x \in S}(\nu(x) + 1)$, since for each $x \in S$ there are $\nu(x) + 1$ possible values of $\nu'(x)$. It is now clear that a $k$-combination of $S$ with repetition is simply a multiset on $S$ with $k$ elements.

Although the reader may be unaware of it, we have already evaluated the number $\left(\!\binom{n}{k}\!\right)$. If $S = \{y_1, \ldots, y_n\}$ and we set $x_i = \nu(y_i)$, then we see that $\left(\!\binom{n}{k}\!\right)$ is the number of solutions in nonnegative integers to $x_1 + x_2 + \cdots + x_n = k$, which we have seen is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

There are two elegant direct combinatorial proofs that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$. For the first, let $1 \leq a_1 < a_2 < \cdots < a_k \leq n + k - 1$ be a $k$-subset of $[n + k - 1]$. Let $b_i = a_i - i + 1$. Then $\{b_1, b_2, \ldots, b_k\}$ is a $k$-multiset on $[n]$. Conversely, given a $k$-multiset $1 \leq b_1 \leq b_2 \leq \cdots \leq b_k \leq n$ on $[n]$, then defining $a_i = b_i + i - 1$ we see that $\{a_1, a_2, \ldots, a_k\}$ is a $k$-subset of $[n + k - 1]$. Hence we have defined a bijection between $\left(\!\binom{[n]}{k}\!\right)$ and $\binom{[n+k-1]}{k}$, as desired. This proof illustrates the technique of *compression*, where we convert a strictly increasing sequence to a weakly increasing sequence.

Our second direct proof that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ is a "geometric" (or "balls into boxes" or "stars and bars") proof, analogous to the proof above that there are $\binom{n-1}{k-1}$ $k$-compositions of $n$. There are $\binom{n+k-1}{k}$ sequences consisting of $k$ dots and $n-1$ vertical bars. An example of such a sequence for $k = 5$ and $n = 7$ is given by

$$||\cdot\ \cdot|\cdot|||\cdot\ \cdot$$

The $n - 1$ bars divide the $k$ dots into $n$ compartments. Let the number of dots in the $i$th compartment be $\nu(i)$. In this way the diagrams correspond to $k$-multisets on $[n]$, so $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$. For the example above, the multiset is $\{3, 3, 4, 7, 7\}$.

The generating function approach to multisets is instructive. In exact analogy to our treatment of subsets of a set $S = \{x_1, \ldots, x_n\}$, we have

$$(1 + x_1 + x_1^2 + \cdots)(1 + x_2 + x_2^2 + \cdots) \cdots (1 + x_n + x_n^2 + \cdots) = \sum_{M=(S,\nu)} \prod_{x_i \in S} x_i^{\nu(x_i)},$$

where the sum is over all finite multisets $M$ on $S$. Put each $x_i = x$. We get

$$\begin{aligned}(1 + x + x^2 + \cdots)^n &= \sum_{M=(S,\nu)} x^{\nu(x_1)+\cdots+\nu(x_n)} \\ &= \sum_{M=(S,\nu)} x^{\#M} \\ &= \sum_{k\geq 0} \left(\!\binom{n}{k}\!\right) x^k.\end{aligned}$$

But

$$(1 + x + x^2 + \cdots)^n = (1 - x)^{-n} = \sum_{k\geq 0} \binom{-n}{k} (-1)^k x^k, \qquad (1.20)$$

so $\left(\!\binom{n}{k}\!\right) = (-1)^k \binom{-n}{k} = \binom{n+k-1}{k}$. The elegant formula

$$\left(\!\binom{n}{k}\!\right) = (-1)^k \binom{-n}{k} \qquad (1.21)$$

is no accident; it is the simplest instance of a *combinatorial reciprocity theorem*. A poset generalization appears in Section 3.15.3, while a more general theory of such results is given in Chapter 4.

The binomial coefficient $\binom{n}{k}$ may be interpreted in the following manner. Each element of an $n$-set $S$ is placed into one of two categories, with $k$ elements in Category 1 and $n-k$ elements in Category 2. (The elements of Category 1 form a $k$-subset $T$ of $S$.) This suggests a generalization allowing more than two categories. Let $(a_1, a_2, \ldots, a_m)$ be a sequence of nonnegative integers summing to $n$, and suppose that we have $m$ categories $C_1, \ldots, C_m$. Let $\binom{n}{a_1,a_2,\ldots,a_m}$ denote the number of ways of assigning each element of an $n$-set $S$ to one of the categories $C_1, \ldots, C_m$ so that exactly $a_i$ elements are assigned to $C_i$. The notation is somewhat at variance with the notation for binomial coefficients (the case $m = 2$), but no confusion should result when we write $\binom{n}{k}$ instead of $\binom{n}{k,n-k}$. The number $\binom{n}{a_1,a_2,\ldots,a_m}$ is called a *multinomial coefficient*. It is customary to regard the elements of $S$ as being $n$ distinguishable balls and the categories as being $m$ distinguishable boxes. Then $\binom{n}{a_1,a_2,\ldots,a_m}$ is the number of ways to place the balls into the boxes so that the $i$th box contains $a_i$ balls.

The multinomial coefficient can also be interpreted in terms of "permutations of a multiset." If $S$ is an $n$-set, then a *permutation* $w$ of $S$ can be defined as a linear ordering $w_1, w_2, \ldots, w_n$ of the elements of $S$. Think of $w$ as a *word* $w_1 w_2 \cdots w_n$ in the alphabet $S$. If $S = \{x_1, \ldots, x_n\}$, then such a word corresponds to the bijection $w : S \to S$ given by $w(x_i) = w_i$, so that a permutation of $S$ may also be regarded as a bijection $S \to S$. Much interesting combinatorics is based on these two different ways of representing permutations; a good example is the second proof of Proposition 5.3.2.

We write $\mathfrak{S}_S$ for the set of permutations of $S$. If $S = [n]$ then we write $\mathfrak{S}_n$ for $\mathfrak{S}_{[n]}$. Since we choose $w_1$ in $n$ ways, then $w_2$ in $n-1$ ways, and so on, we clearly have $\#\mathfrak{S}_S = n!$. In an analogous manner we can define a permutation $w$ of a multiset $M$ of cardinality $n$ to be a linear ordering $w_1, w_2, \ldots, w_n$ of the "elements" of $M$; that is, if $M = (S, \nu)$ then the element $x \in S$ appears exactly $\nu(x)$ times in the permutation. Again we think of $w$ as a word $w_1 w_2 \cdots w_n$. For instance, there are 12 permutations of the multiset $\{1, 1, 2, 3\}$; namely, 1123, 1132, 1213, 1312, 1231, 1321, 2113, 2131, 2311, 3112, 3121, 3211. Let $\mathfrak{S}_M$ denote the set of permutations of $M$. If $M = \{x_1^{a_1}, \ldots, x_m^{a_m}\}$ and $\#M = n$, then it is clear that

$$\#\mathfrak{S}_M = \binom{n}{a_1, a_2, \ldots, a_m}. \tag{1.22}$$

Indeed, if $x_i$ appears in position $j$ of the permutation, then we put the element $j$ of $[n]$ into Category $i$.

Our results on binomial coefficients extend straightforwardly to multinomial coefficients. In particular, we have

$$\binom{n}{a_1, a_2, \ldots, a_m} = \frac{n!}{a_1!\, a_2! \cdots a_m!}. \tag{1.23}$$

Among the many ways to prove this result, we can place $a_1$ elements of $S$ into Category 1 in $\binom{n}{a_1}$ ways, then $a_2$ of the remaining $n - a_1$ elements of $[n]$ into Category 2 in $\binom{n-a_1}{a_2}$ ways,

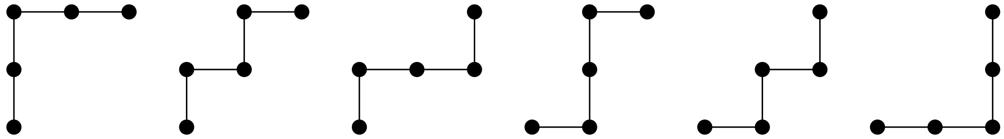

Figure 1.1: Six lattice paths

etc., yielding

$$\binom{n}{a_1, a_2, \ldots, a_m} = \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\cdots-a_{m-1}}{a_m} = \frac{n!}{a_1!\,a_2!\cdots a_m!}. \tag{1.24}$$

Equation (1.24) is often a useful device for reducing problems on multinomial coefficients to binomial coefficients. We leave to the reader the (easy) multinomial analogue (known as the *multinomial theorem*) of equation (1.18), namely,

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{a_1+\cdots+a_m=n} \binom{n}{a_1, a_2, \ldots, a_m} x_1^{a_1} \cdots x_m^{a_m},$$

where the sum ranges over all $(a_1, \ldots, a_m) \in \mathbb{N}^m$ satisfying $a_1 + \cdots + a_m = n$. Note that $\binom{n}{1,1,\ldots,1} = n!$, the number of permutations of an $n$-element set.

Binomials and multinomial coefficients have an important geometric interpretation in terms of lattice paths. Let $S$ be a subset of $\mathbb{Z}^d$. More generally, we could replace $\mathbb{Z}^d$ by any lattice (discrete subgroup of full rank) in $\mathbb{R}^d$, but for simplicity we consider only $\mathbb{Z}^d$. A *lattice path* $L$ in $\mathbb{Z}^d$ of length $k$ with steps in $S$ is a sequence $v_0, v_1, \ldots, v_k \in \mathbb{Z}^d$ such that each consecutive difference $v_i - v_{i-1}$ lies in $S$. We say that $L$ *starts at* $v_0$ and *ends at* $v_k$, or more simply that $L$ goes *from* $v_0$ *to* $v_k$. Figure 1.1 shows the six lattice paths in $\mathbb{Z}^2$ from $(0, 0)$ to $(2, 2)$ with steps $(1, 0)$ and $(0, 1)$.

**1.2.1 Proposition.** *Let* $v = (a_1, \ldots, a_d) \in \mathbb{N}^d$*, and let* $e_i$ *denote the* $i$*th unit coordinate vector in* $\mathbb{Z}^d$*. The number of lattice paths in* $\mathbb{Z}^d$ *from the origin* $(0, 0, \ldots, 0)$ *to* $v$ *with steps* $e_1, \ldots, e_d$ *is given by the multinomial coefficient* $\binom{a_1+\cdots+a_d}{a_1,\ldots,a_d}$.

*Proof.* Let $v_0, v_1, \ldots, v_k$ be a lattice path being counted. Then the sequence $v_1 - v_0, v_2 - v_1, \ldots, v_k - v_{k-1}$ is simply a sequence consisting of $a_i$ $e_i$'s in some order. The proof follows from equation (1.22). □

Proposition 1.2.1 is the most basic result in the vast subject of *lattice path enumeration*. Further results in this area will appear throughout this book.