1.3 Cycles and Inversions

Permutations of sets and multisets are among the richest objects in enumerative combinatorics. A basic reason for this fact is the wide variety of ways to *represent* a permutation combinatorially. We have already seen that we can represent a set permutation either as a *word* or a *function*. In fact, for any set S the function $w : [n] \to S$ given by $w(i) = w_i$ corresponds to the word $w_1 w_2 \cdots w_n$. Several additional representations will arise in Section 1.5. Many of the basic results derived here will play an important role in later analysis of more complicated objects related to permutations.

A second reason for the richness of the theory of permutations is the wide variety of interesting "statistics" of permutations. In the broadest sense, a statistic on some class $\mathcal{C}$ of combinatorial objects is just a function $f : \mathcal{C} \to S$, where S is any set (often taken to be $\mathbb{N}$). We want $f(x)$ to capture some combinatorially interesting feature of x. For instance, if x is a (finite) set, then $f(x)$ could be its number of elements. We can think of f as *refining* the enumeration of objects in $\mathcal{C}$. For instance, if $\mathcal{C}$ consists of all subsets of an n-set S and $f(x) = \#x$, then f refines the number 2^n of subsets of S into a sum $2^n = \sum_k \binom{n}{k}$, where $\binom{n}{k}$ is the number of subsets of S with k elements. In this section and the next two we will discuss a number of different statistics on permutations.

Cycle Structure

If we regard a set permutation w as a bijection $w : S \to S$, then it is natural to consider for each $x \in S$ the sequence $x, w(x), w^2(x), \ldots$. Eventually (since w is a bijection and S is assumed finite) we must return to x. Thus for some unique $\ell \geq 1$ we have that $w^\ell(x) = x$ and that the elements $x, w(x), \ldots, w^{\ell-1}(x)$ are distinct. We call the sequence $(x, w(x), \ldots, w^{\ell-1}(x))$ a *cycle* of w of length ℓ. The cycles $(x, w(x), \ldots, w^{\ell-1}(x))$ and $(w^i(x), w^{i+1}(x), \ldots, w^{\ell-1}(x), x, \ldots, w^{i-1}(x))$ are considered the same. Every element of S then appears in a unique cycle of w, and we may regard w as a disjoint union or *product* of its distinct cycles $C_1, \ldots, C_k$, written $w = C_1 \cdots C_k$. For instance, if $w : [7] \to [7]$ is defined by $w(1) = 4$, $w(2) = 2$, $w(3) = 7$, $w(4) = 1$, $w(5) = 3$, $w(6) = 6$, $w(7) = 5$ (or $w = 4271365$ as a word), then $w = (14)(2)(375)(6)$. Of course this representation of w in disjoint cycle notation is not unique; we also have for instance $w = (753)(14)(6)(2)$.

A geometric or graphical representation of a permutation w is often useful. A finite *directed graph* or *digraph* D is a triple (V, E, ϕ), where $V = \{x_1, \ldots, x_n\}$ is a set of *vertices*, E is a finite set of (directed) *edges* or *arcs*, and ϕ is a map from E to $V \times V$. If ϕ is injective then we call D a *simple* digraph, and we can think of E as a subset of $V \times V$. If e is an edge with $\phi(e) = (x, y)$, then we represent e as an arrow directed from x to y. If w is permutation of the set S, then define the *digraph* D_w *of* w to be the directed graph with vertex set S and edge set $\{(x, y) : w(x) = y\}$. In other words, for every vertex x there is an edge from x to $w(x)$. Digraphs of permutations are characterized by the property that every vertex has one edge pointing out and one pointing in. The disjoint cycle decomposition of a permutation of a finite set guarantees that D_w will be a disjoint union of directed cycles. For instance, Figure 1.2 shows the digraph of the permutation $w = (14)(2)(375)(6)$.

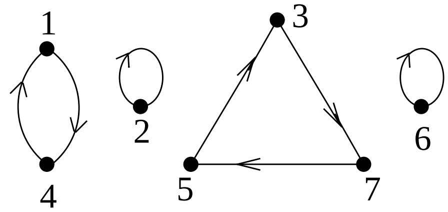

Figure 1.2: The digraph of the permutation (14)(2)(375)(6)

We noted above that the disjoint cycle notation of a permutation is not unique. We can define a *standard representation* by requiring that (a) each cycle is written with its largest element first, and (b) the cycles are written in increasing order of their largest element. Thus the standard form of the permutation $w = (14)(2)(375)(6)$ is $(2)(41)(6)(753)$. Define $\widehat{w}$ to be the word (or permutation) obtained from w by writing it in standard form and erasing the parentheses. For example, with $w = (2)(41)(6)(753)$ we have $\widehat{w} = 2416753$. Now observe that we can uniquely recover w from $\widehat{w}$ by inserting a left parenthesis in $\widehat{w} = a_1 a_2 \cdots a_n$ preceding every *left-to-right maximum* or *record* (also called *outstanding element*); that is, an element a_i such that $a_i > a_j$ for every $j < i$. Then insert a right parenthesis where appropriate; that is, before every internal left parenthesis and at the end. Thus the map $w \mapsto \widehat{w}$ is a *bijection* from $\mathfrak{S}_n$ to itself, known as the *fundamental bijection*. Let us sum up this information as a proposition.

1.3.1 Proposition. *(a) The map* $\mathfrak{S}_n \xrightarrow{\wedge} \mathfrak{S}_n$ *defined above is a bijection.*

(b) If $w \in \mathfrak{S}_n$ *has* k *cycles, then* $\widehat{w}$ *has* k *left-to-right maxima.*

If $w \in \mathfrak{S}_S$ where $\#S = n$, then let $c_i = c_i(w)$ be the number of cycles of w of length i. Note that $n = \sum i c_i$. Define the *type* of w, denoted $\mathrm{type}(w)$, to be the sequence $(c_1, \dots, c_n)$. The total number of cycles of w is denoted $c(w)$, so $c(w) = c_1(w) + \cdots + c_n(w)$.

1.3.2 Proposition. *The number of permutations* $w \in \mathfrak{S}_S$ *of type* $(c_1, \dots, c_n)$ *is equal to* $n!/1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!$.

Proof. Let $w = w_1 w_2 \cdots w_n$ be any permutation of S. Parenthesize the word w so that the first c_i cycles have length 1, the next c_2 have length 2, and so on. For instance, if $(c_1, \dots, c_9) = (1, 2, 0, 1, 0, 0, 0, 0, 0)$ and $w = 427619583$, then we obtain $(4)(27)(61)(9583)$. In general we obtain the disjoint cycle decomposition of a permutation w' of type $(c_1, \dots, c_n)$. Hence we have defined a map $\Phi : \mathfrak{S}_S \to \mathfrak{S}_S^{\boldsymbol{c}}$, where $\mathfrak{S}_S^{\boldsymbol{c}}$ is the set of all $u \in \mathfrak{S}_S$ of type $\boldsymbol{c} = (c_1, \dots, c_n)$. Given $u \in \mathfrak{S}_S^{\boldsymbol{c}}$, we claim that there are $1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!$ ways to write it in disjoint cycle notation so that the cycle lengths are weakly increasing from left to right. Namely, order the cycles of length i in $c_i!$ ways, and choose the first elements of these cycles in i^{c_i} ways. These choices are all independent, so the claim is proved. Hence for each $u \in \mathfrak{S}_S^{\boldsymbol{c}}$ we have $\#\Phi^{-1}(u) = 1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!$, and the proof follows since $\#\mathfrak{S}_S = n!$. □

Note. The proof of Proposition 1.3.2 can easily be converted into a bijective proof of the identity

$$n! = 1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!\left(\#\mathfrak{S}_S^{\boldsymbol{c}}\right),$$

analogous to our bijective proof of equation (1.16).

Proposition 1.3.2 has an elegant and useful formulation in terms of generating functions. Suppose that $w \in \mathfrak{S}_n$ has type $(c_1, \ldots, c_n)$. Write

$$t^{\mathrm{type}(w)} = t_1^{c_1} t_2^{c_2} \cdots t_n^{c_n},$$

and define the *cycle indicator* or *cycle index* of $\mathfrak{S}_n$ to be the polynomial

$$Z_n = Z_n(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} t^{\mathrm{type}(w)}. \tag{1.25}$$

(Set $Z_0 = 1$.) For instance,

$$\begin{aligned}
Z_1 &= t_1 \\
Z_2 &= \frac{1}{2}(t_1^2 + t_2) \\
Z_3 &= \frac{1}{6}(t_1^3 + 3t_1t_2 + 2t_3) \\
Z_4 &= \frac{1}{24}(t_1^4 + 6t_1^2t_2 + 8t_1t_3 + 3t_2^2 + 6t_4).
\end{aligned}$$

1.3.3 Theorem. *We have*

$$\sum_{n \geq 0} Z_n x^n = \exp\left(t_1 x + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots\right). \tag{1.26}$$

Proof. We give a naive computational proof. For a more conceptual proof, see Example 5.2.10. Let us expand the right-hand side of equation (1.26):

$$\begin{aligned}
\exp\left(\sum_{i \geq 1} t_i \frac{x^i}{i}\right) &= \prod_{i \geq 1} \exp\left(t_i \frac{x^i}{i}\right) \\
&= \prod_{i \geq 1} \sum_{j \geq 0} t_i^j \frac{x^{ij}}{i^j j!}.
\end{aligned} \tag{1.27}$$

Hence the coefficient of $t_1^{c_1} \cdots t_n^{c_n} x^n$ is equal to 0 unless $\sum ic_i = n$, in which case it is equal to

$$\frac{1}{1^{c_1} c_1! \, 2^{c_2} c_2! \cdots} = \frac{1}{n!} \frac{n!}{1^{c_1} c_1! \, 2^{c_2} c_2! \cdots}.$$

Comparing with Proposition 1.3.2 completes the proof. $\square$

Let us give two simple examples of the use of Theorem 1.3.3. For some additional examples, see Exercises 5.10 and 5.11. A more general theory of cycle indicators based on symmetric functions is given in Section 7.24. Write $F(t; x) = F(t_1, t_2, \ldots; x)$ for the right-hand side of equation (1.26).

1.3.4 Example. Let $e_6(n)$ be the number of permutations $w \in \mathfrak{S}_n$ satisfying $w^6 = 1$. A permutation w satisfies $w^6 = 1$ if and only if all its cycles have length 1,2,3 or 6. Hence

$$e_6(n) = n!\, Z_n(t_i = 1 \text{ if } i|6,\ t_i = 0 \text{ otherwise}).$$

There follows

$$\begin{aligned}\sum_{n\geq 0} e_6(n)\frac{x^n}{n!} &= F(t_i = 1 \text{ if } i|6,\ t_i = 0 \text{ otherwise})\\ &= \exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^6}{6}\right).\end{aligned}$$

For the obvious generalization to permutations w satisfying $w^r = 1$, see equation (5.31).

1.3.5 Example. Let $E_k(n)$ denote the expected number of k-cycles in a permutation $w \in \mathfrak{S}_n$. It is understood that the expectation is taken with respect to the uniform distribution on $\mathfrak{S}_n$, so

$$E_k(n) = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} c_k(w),$$

where $c_k(w)$ denotes the number of k-cycles in w. Now note that from the definition (1.25) of Z_n we have

$$E_k(n) = \frac{\partial}{\partial t_k} Z_n(t_1,\ldots,t_n)|_{t_i=1}.$$

Hence

$$\begin{aligned}\sum_{n\geq 0} E_k(n)x^n &= \frac{\partial}{\partial t_k}\exp\left(t_1 x + t_2\frac{x^2}{2} + t_3\frac{x^3}{3} + \cdots\right)\Big|_{t_i=1}\\ &= \frac{x^k}{k}\exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)\\ &= \frac{x^k}{k}\exp\log(1-x)^{-1}\\ &= \frac{x^k}{k}\frac{1}{1-x}\\ &= \frac{x^k}{k}\sum_{n\geq 0} x^n.\end{aligned}$$

It follows that $E_k(n) = 1/k$ for $n \geq k$. Can the reader think of a simple explanation (Exercise 1.120)?

Now define $c(n,k)$ to be the number of permutations $w \in \mathfrak{S}_n$ with exactly k cycles. The number $s(n,k) := (-1)^{n-k}c(n,k)$ is known as a *Stirling number of the first kind*, and $c(n,k)$ is called a *signless Stirling number of the first kind.*

1.3.6 Lemma. *The numbers $c(n,k)$ satisfy the recurrence*

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1), \quad n,k \geq 1,$$

with the initial conditions $c(n,k) = 0$ if $n < k$ or $k = 0$, except $c(0,0) = 1$.

Proof. Choose a permutation $w \in \mathfrak{S}_{n-1}$ with k cycles. We can insert the symbol n after any of the numbers $1, 2, \ldots, n-1$ in the disjoint cycle decomposition of w in $n-1$ ways, yielding the disjoint cycle decomposition of a permutation $w' \in \mathfrak{S}_n$ with k cycles for which n appears in a cycle of length at least 2. Hence there are $(n-1)c(n-1,k)$ permutations $w' \in \mathfrak{S}_n$ with k cycles for which $w'(n) \neq n$.

On the other hand, if we choose a permutation $w \in \mathfrak{S}_{n-1}$ with $k-1$ cycles we can extend it to a permutation $w' \in \mathfrak{S}_n$ with k cycles satisfying $w'(n) = n$ by defining

$$w'(i) = \begin{cases} w(i), & \text{if } i \in [n-1] \\ n, & \text{if } i = n. \end{cases}$$

Thus there are $c(n-1,k-1)$ permutations $w' \in \mathfrak{S}_n$ with k cycles for which $w'(n) = n$, and the proof follows. $\square$

Most of the elementary properties of the numbers $c(n,k)$ can be established using Lemma 1.3.6 together with mathematical induction. However, combinatorial proofs are to be preferred whenever possible. An illuminating illustration of the various techniques available to prove elementary combinatorial identities is provided by the next result.

1.3.7 Proposition. *Let t be an indeterminate and fix $n \geq 0$. Then*

$$\sum_{k=0}^{n} c(n,k)t^k = t(t+1)(t+2)\cdots(t+n-1). \tag{1.28}$$

First proof. This proof may be regarded as "semi-combinatorial" since it is based directly on Lemma 1.3.6, which had a combinatorial proof. Let

$$F_n(t) = t(t+1)\cdots(t+n-1) = \sum_{k=0}^{n} b(n,k)t^k.$$

Clearly $b(n,k) = 0$ if $n = 0$ or $k = 0$, except $b(0,0) = 1$ (an empty product is equal to 1). Moreover, since

$$\begin{aligned} F_n(t) &= (t+n-1)F_{n-1}(t) \\ &= \sum_{k=1}^{n} b(n-1,k-1)t^k + (n-1)\sum_{k=0}^{n-1} b(n-1,k)t^k, \end{aligned}$$

there follows $b(n,k) = (n-1)b(n-1,k) + b(n-1,k-1)$. Hence $b(n,k)$ satisfies the same recurrence and initial conditions as $c(n,k)$, so they agree. $\square$

Second proof. Our next proof is a straightforward argument using generating functions. In terms of the cycle indicator Z_n we have

$$\sum_{k=0}^{n} c(n,k)t^k = n!Z_n(t,t,t,\ldots).$$

Hence substituting $t_i = t$ in equation (1.26) gives

$$\begin{aligned}
\sum_{n\geq 0}\sum_{k=0}^{n} c(n,k)t^k\frac{x^n}{n!} &= \exp t(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots) \\
&= \exp t(\log(1-x)^{-1}) \\
&= (1-x)^{-t} \\
&= \sum_{n\geq 0}(-1)^n\binom{-t}{n}x^n \\
&= \sum_{n\geq 0} t(t+1)\ldots(t+n-1)\frac{x^n}{n!},
\end{aligned}$$

and the proof follows from taking coefficient of $x^n/n!$. $\square$

Third proof. The coefficient of t^k in $F_n(t)$ is

$$\sum_{1\leq a_1<a_2<\cdots<a_{n-k}\leq n-1} a_1a_2\cdots a_{n-k}, \tag{1.29}$$

where the sum is over all $\binom{n-1}{n-k}$ $(n-k)$-subsets $\{a_1,\ldots,a_{n-k}\}$ of $[n-1]$. (Though irrelevant here, it is interesting to note that this sum is just the $(n-k)$th elementary symmetric function of $1,2,\ldots,n-1$.) Clearly (1.29) counts the number of pairs (S,f), where $S\in\binom{[n-1]}{n-k}$ and $f : S\to[n-1]$ satisfies $f(i)\leq i$. Thus we seek a bijection $\phi:\Omega\to\mathfrak{S}_{n,k}$ between the set Ω of all such pairs (S,f), and the set $\mathfrak{S}_{n,k}$ of $w\in\mathfrak{S}_n$ with k cycles.

Given $(S,f)\in\Omega$ where $S=\{a_1,\ldots,a_{n-k}\}_<\subseteq[n-1]$, define $T=\{j\in[n] : n-j\notin S\}$. Let the elements of $[n]-T$ be $b_1>b_2>\cdots>b_{n-k}$. Define $w=\phi(S,f)$ to be that permutation that when written in standard form satisfies: (i) the first (=greatest) elements of the cycles of w are the elements of T, and (ii) for $i\in[n-k]$ the number of elements of w preceding b_i and larger than b_i is $f(a_i)$. We leave it to the reader to verify that this construction yields the desired bijection. $\square$

1.3.8 Example. Suppose that in the above proof $n=9$, $k=4$, $S=\{1,3,4,6,8\}$, $f(1)=1$, $f(3)=2$, $f(4)=1$, $f(6)=3$, $f(8)=6$. Then $T=\{2,4,7,9\}$, $[9]-T=\{1,3,5,6,8\}$, and $w=(2)(4)(753)(9168)$.

Fourth proof of Proposition 1.3.7. There are two basic ways of giving a combinatorial proof that two polynomials are equal: (i) showing that their coefficients are equal, and (ii) showing that they agree for sufficiently many values of their variable(s). We have already established Proposition 1.3.7 by the first technique; here we apply the second. If two polynomials in a single variable t (over the complex numbers, say) agree for all $t\in\mathbb{P}$, then they agree as polynomials. Thus it suffices to establish (1.28) for all $t\in\mathbb{P}$.

Let $t\in\mathbb{P}$, and let $C(w)$ denote the set of cycles of $w\in\mathfrak{S}_n$. The left-hand side of (1.28) counts all pairs (w,f), where $w\in\mathfrak{S}_n$ and $f : C(w)\to[t]$. The right-hand side counts integer sequences $(a_1,a_2,\ldots,a_n)$ where $0\leq a_i\leq t+n-i-1$. (There are historical reasons for this restriction of a_i, rather than, say, $1 \le a_i \le t+i-1$.) Given such a sequence $(a_1, a_2, \ldots, a_n)$, the following simple algorithm may be used to define (w, f). First write down the number n and regard it as starting a cycle C_1 of w. Let $f(C_1) = a_n + 1$. Assuming $n, n-1, \ldots, n-i+1$ have been inserted into the disjoint cycle notation for w, we now have two possibilities:

i. $0 \le a_{n-i} \le t-1$. Then start a new cycle C_j with the element $n-i$ to the left of the previously inserted elements, and set $f(C_j) = a_{n-i} + 1$.

ii. $a_{n-i} = t+k$ where $0 \le k \le i-1$. Then insert $n-i$ into an old cycle so that it is not the leftmost element of any cycle, and so that it appears to the right of $k+1$ of the numbers previously inserted.

This procedure establishes the desired bijection. □

1.3.9 Example. Suppose $n = 9$, $t = 4$, and $(a_1, \ldots, a_9) = (4, 8, 5, 0, 7, 5, 2, 4, 1)$. Then w is built up as follows:

$$\begin{array}{l}
(9) \\
(98) \\
(7)(98) \\
(7)(968) \\
(7)(9685) \\
(4)(7)(9685) \\
(4)(73)(9685) \\
(4)(73)(96285) \\
(41)(73)(96285).
\end{array}$$

Moreover, $f(96285) = 2$, $f(73) = 3$, $f(41) = 1$.

Note that if we set $t = 1$ in the preceding proof, we obtain a combinatorial proof of the following result.

1.3.10 Proposition. *Let $n, k \in \mathbb{P}$. The number of integer sequences $(a_1, \ldots, a_n)$ such that $0 \le a_i \le n-i$ and exactly k values of a_i equal 0 is $c(n, k)$*

Note that because of Proposition 1.3.1 we obtain "for free" the enumeration of permutations by left-to-right maxima.

1.3.11 Corollary. *The number of $w \in \mathfrak{S}_n$ with k left-to-right maxima is $c(n, k)$.*

Corollary 1.3.11 illustrates one benefit of having different ways of representing the same object (here a permutation)—different enumerative problems involving the object turn out to be equivalent.

Inversions

The fourth proof of Proposition 1.3.7 (in the case $t = 1$) associated a permutation $w \in \mathfrak{S}_n$ with an integer sequence $(a_1, \ldots, a_n)$, $0 \le a_i \le n-i$. There is a different method for accomplishing this which is perhaps more natural. Given such a vector $(a_1, \dots, a_n)$, assume that $n, n-1, \dots, n-i+1$ have been inserted into w, expressed this time as a *word* (rather than a product of cycles). Then insert $n-i$ so that it has a_{n-i} elements to its left. For example, if $(a_1, \dots, a_9) = (1, 5, 2, 0, 4, 2, 0, 1, 0)$, then w is built up as follows:

$$\begin{array}{l} 9 \\ 98 \\ 798 \\ 7968 \\ 79685 \\ 479685 \\ 4739685 \\ 47396285 \\ 417396285. \end{array}$$

Clearly a_i is the number of entries j of w to the left of i satisfying $j > i$. A pair (w_i, w_j) is called an *inversion* of the permutation $w = w_1 w_2 \cdots w_n$ if $i < j$ and $w_i > w_j$. The above sequence $I(w) = (a_1, \dots, a_n)$ is called the *inversion table* of w. The above algorithm for constructing w from its inversion table $I(w)$ establishes the following result.

1.3.12 Proposition. *Let*

$$\mathcal{T}_n = \{(a_1, \dots, a_n) \,:\, 0 \le a_i \le n-i\} = [0, n-1] \times [0, n-2] \times \cdots \times [0, 0].$$

The map $I : \mathfrak{S}_n \to \mathcal{T}_n$ that sends each permutation to its inversion table is a bijection.

Therefore, the inversion table $I(w)$ is yet another way to represent a permutation w. Let us also mention that the *code* of a permutation w is defined by $\mathrm{code}(w) = I(w^{-1})$. Equivalently, if $w = w_1 \cdots w_n$ and $\mathrm{code}(w) = (c_1, \dots, c_n)$, then c_i is equal to the number of elements w_j to the right of w_i (i.e., $i < j$) such that $w_i > w_j$. The question of whether to use $I(w)$ or $\mathrm{code}(w)$ depends on the problem at hand and is clearly only a matter of convenience. Often it makes no difference which is used, such as in obtaining the next corollary.

1.3.13 Corollary. *Let* $\mathrm{inv}(w)$ *denote the number of inversions of the permutation* $w \in \mathfrak{S}_n$. *Then*

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}). \tag{1.30}$$

Proof. If $I(w) = (a_1, \dots, a_n)$ then $\mathrm{inv}(w) = a_1 + \cdots + a_n$. hence

$$\begin{aligned} \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} &= \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \cdots \sum_{a_n=0}^{0} q^{a_1+a_2+\cdots+a_n} \\ &= \left(\sum_{a_1=0}^{n-1} q^{a_1} \right) \left(\sum_{a_2=0}^{n-2} q^{a_2} \right) \cdots \left(\sum_{a_n=0}^{0} q^{a_n} \right), \end{aligned}$$

as desired. $\square$

The polynomial $(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$ is called "the q-analogue of $n!$" and is denoted $(\boldsymbol{n})!$. Moreover, we denote the polynomial $1+q+\cdots+q^{n-1} = (1-q^n)/(1-q)$ by $(\boldsymbol{n})$ and call it "the q-analogue of n," so that

$$(\boldsymbol{n})! = (\mathbf{1})(\mathbf{2})\cdots(\boldsymbol{n}).$$

In general, a *q-analogue* of a mathematical object is an object depending on the variable q that "reduces to" (an admittedly vague term) the original object when we set $q = 1$. To be a "satisfactory" q-analogue more is required, but there is no precise definition of what is meant by "satisfactory." Certainly one desirable property is that the original object concerns finite sets, while the q-analogue can be interpreted in terms of subspaces of finite-dimensional vector spaces over the finite field $\mathbb{F}_q$. For instance, $n!$ is the number of sequences $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = [n]$ of subsets of $[n]$. (The symbol $\subset$ denotes strict inclusion, so $\#S_i = i$.) Similarly if q is a prime power then $(\boldsymbol{n})!$ is the number of sequences $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{F}_q^n$ of subspaces of the n-dimensional vector space $\mathbb{F}_q^n$ over $\mathbb{F}_q$ (so $\dim V_i = i$). For this reason $(\boldsymbol{n})!$ is regarded as a satisfactory q-analogue of $n!$. We can also regard an i-dimensional vector space over $\mathbb{F}_q$ as the q-analogue of an i-element set. Many more instances of q-analogues will appear throughout this book, especially in Section 1.10. The theory of binomial posets developed in Section 3.18 gives a partial explanation for the existence of certain classes of q-analogues including $(\boldsymbol{n})!$.

We conclude this section with a simple but important property of the statistic inv.

1.3.14 Proposition. *For any* $w = w_1w_2\cdots w_n \in \mathfrak{S}_n$ *we have* $\mathrm{inv}(w) = \mathrm{inv}(w^{-1})$.

Proof. The pair (i,j) is an inversion of w if and only if (w_j, w_i) is an inversion of w^{-1}. $\square$