A Note about the Exercises

Each exercise is given a difficulty rating, as follows.

1. routine, straightforward
2. somewhat difficult or tricky
3. difficult
4. horrendously difficult
5. unsolved

Further gradations are indicated by + and −. Thus [1–] denotes an utterly trivial problem, and [5–] denotes an unsolved problem that has received little attention and may not be too difficult. A rating of [2+] denotes about the hardest problem that could be reasonably assigned to a class of graduate students. A few students may be capable of solving a [3–] problem, while almost none could solve a [3] in a reasonable period of time. Of course the ratings are subjective, and there is always the possibility of an overlooked simple proof that would lower the rating. Some problems (seemingly) require results or techniques from other branches of mathematics that are not usually associated with combinatorics. Here the rating is less meaningful—it is based on an assessment of how likely the reader is to discover for herself or himself the relevance of these outside techniques and results. An asterisk after the difficulty rating indicates that no solution is provided.

EXERCISES FOR CHAPTER 1

1. [1–] Let S and T be disjoint one-element sets. Find the number of elements of their union $S \cup T$.

2. [1+] We continue with a dozen simple numerical problems. Find as simple a solution as possible.

   (a) How many subsets of the set $[10] = \{1, 2, \ldots, 10\}$ contain at least one odd integer?

   (b) In how many ways can seven people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (not necessarily on the same side)?

   (c) How many permutations $w : [6] \to [6]$ satisfy $w(1) \neq 2$?

   (d) How many permutations of [6] have exactly two cycles (i.e., find $c(6, 2)$)?

   (e) How many partitions of [6] have exactly three blocks (i.e., find $S(6, 3)$)?

   (f) There are four men and six women. Each man marries one of the women. In how many ways can this be done?

(g) Ten people split up into five groups of two each. In how many ways can this be done?

(h) How many compositions of 19 use only the parts 2 and 3?

(i) In how many different ways can the letters of the word MISSISSIPPI be arranged if the four S's cannot appear consecutively?

(j) How many sequences $(a_1, a_2, \ldots, a_{12})$ are there consisting of four 0's and eight 1's, if no two consecutive terms are both 0's?

(k) A box is filled with three blue socks, three red socks, and four chartreuse socks. Eight socks are pulled out, one at a time. In how many ways can this be done? (Socks of the same color are indistinguishable.)

(l) How many functions $f : [5] \to [5]$ are at most two-to-one, i.e., $\#f^{-1}(n) \leq 2$ for all $n \in [5]$?

3. Give *combinatorial* proofs of the following identities, where x, y, n, a, b are nonnegative integers.

(a) [2–] $\displaystyle\sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n}$

(b) [1+] $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

(c) [3] $\displaystyle\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n$

(d) [3–] $\displaystyle\sum_{k=0}^{m} \binom{x+y+k}{k}\binom{y}{a-k}\binom{x}{b-k} = \binom{x+a}{b}\binom{y+b}{a}$, where $m = \min(a, b)$

(e) [1] $2\dbinom{2n-1}{n} = \dbinom{2n}{n}$

(f) [2–] $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0,\ n \geq 1$

(g) [2+] $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{j=0}^{n} \binom{n}{j}\binom{2n-j}{n}(x-1)^j$

(h) [3–] $\displaystyle\sum_{i+j+k=n} \binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k} = \sum_{r=0}^{n} \binom{2r}{r}$, where $i, j, k \in \mathbb{N}$

4. [2]* Fix $j, k \in \mathbb{Z}$. Show that

$$\sum_{n\geq 0} \frac{(2n-j-k)!x^n}{(n-j)!(n-k)!(n-j-k)!n!} = \left[\sum_{n\geq 0} \frac{x^n}{n!(n-j)!}\right]\left[\sum_{n\geq 0} \frac{x^n}{n!(n-k)!}\right].$$

Any term with $(-r)!$ in the denominator, where $r > 0$, is set equal to 0.

5. [2]* Show that

$$\sum_{n_1,\ldots,n_k\geq 0} \min(n_1,\ldots,n_k) x_1^{n_1}\cdots x_k^{n_k} = \frac{x_1\cdots x_k}{(1-x_1)\cdots(1-x_k)(1-x_1x_2\cdots x_k)}.$$

6. [3–]* For $n\in\mathbb{Z}$ let

$$J_n(2x) = \sum_{k\in\mathbb{Z}} \frac{(-1)^k x^{n+2k}}{k!(n+k)!},$$

where we set $1/j! = 0$ for $j<0$. Show that

$$e^x = \sum_{n\geq 0} L_n J_n(2x),$$

where $L_0 = 1, L_1 = 1,\ L_2 = 3,\ L_{n+1} = L_n + L_{n-1}$ for $n\geq 2$. (The numbers L_n for $n\geq 1$ are *Lucas numbers.*)

7. [2]* Let

$$e^{x+\frac{x^2}{2}} = \sum_{n\geq 0} f(n)\frac{x^n}{n!}.$$

Find a simple expression for $\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f(i)$. (See equation (1.13).)

8. (a) [2–] Show that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0}\binom{2n}{n}x^n.$$

(b) [2–] Find $\sum_{n\geq 0}\binom{2n-1}{n}x^n$.

9. Let $f(m,n)$ be the number of paths from $(0,0)$ to $(m,n)\in\mathbb{N}\times\mathbb{N}$, where each step is of the form $(1,0)$, $(0,1)$, or $(1,1)$.

(a) [1+]* Show that $\sum_{m\geq 0}\sum_{n\geq 0} f(m,n)x^m y^n = (1-x-y-xy)^{-1}$.

(b) [3–] Find a simple explicit expression for $\sum_{n\geq 0} f(n,n)x^n$.

10. [2+] Let $f(n,r,s)$ denote the number of subsets S of $[2n]$ consisting of r odd and s even integers, with no two elements of S differing by 1. Give a bijective proof that $f(n,r,s) = \binom{n-r}{s}\binom{n-s}{r}$.

11. (a) [2+] Let $m,n\in\mathbb{N}$. Interpret the integral

$$B(m+1,n+1) = \int_0^1 u^m(1-u)^n\,du,$$

as a probability and evaluate it by combinatorial reasoning.

(b) [3+] Let $n \in \mathbb{P}$ and $r, s, t \in \mathbb{N}$. Let x, y_k, z_k and a_{ij} be indeterminates, with $1 \leq k \leq n$ and $1 \leq i < j \leq n$. Let M be the multiset with n occurrences of x, r occurrences of each y_k, s occurrences of each z_k, and $2t$ occurrences of each a_{ij}. Let $f(n, r, s, t)$ be the number of permutations w of M such that (i) all y_k's appear before the kth x (reading the x's from left-to-right in w), (ii) all z_k's appear after the kth x, and (iii) all a_{ij}'s appear between the ith x and jth x. Show that

$$f(n, r, s, t) = \frac{[(r+s+1)n + tn(n-1)]!}{n!r!^n s!^n t!^n (2t)!^{\binom{n}{2}}} \cdot \prod_{j=1}^{n} \frac{(r+(j-1)t)!(s+(j-1)t)!(jt)!}{(r+s+1+(n+j-2)t)!}. \qquad (1.119)$$

(c) [3–] Consider the following chess position.

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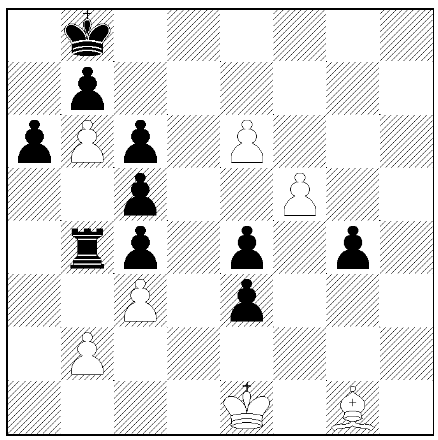


Black is to make 14 consecutive moves, after which White checkmates Black in one move. Black may not move into check, and may not check White (except possibly on his last move). Black and White are *cooperating* to achieve the aim of checkmate. (In chess problem parlance, this problem is called a *serieshelpmate in 14*.) How many different solutions are there?

12. [2+]* Choose n points on the circumference of a circle in "general position." Draw all $\binom{n}{2}$ chords connecting two of the points. ("General position" means that no three of these chords intersect in a point.) Into how many regions will the interior of the circle be divided? Try to give an elegant proof avoiding induction, finite differences, generating functions, summations, etc.

13. [2] Let p be prime and $a \in \mathbb{P}$. Show *combinatorially* that $a^p - a$ is divisible by p. (A combinatorial proof would consist of exhibiting a set S with $a^p - a$ elements and a partition of S into pairwise disjoint subsets, each with p elements.)

14. (a) [2+] Let p be a prime, and let $n = \sum a_i p^i$ and $m = \sum b_i p^i$ be the p-ary expansions of the positive integers m and n. Show that

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots \pmod{p}.$$

(b) [3–] Use (a) to determine when $\binom{n}{m}$ is odd. For what n is $\binom{n}{m}$ odd for all $0 \le m \le n$? In general, how many coefficients of the polynomial $(1+x)^n$ are odd?

(c) [2+] It follows from (a), and is easy to show directly, that $\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$. Give a *combinatorial proof* that in fact $\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p^2}$.

(d) [3–] If $p \ge 5$, then show in fact

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p^3}.$$

Is there a combinatorial proof?

(e) [3–] Give a simple description of the largest power of p dividing $\binom{n}{m}$.

15. (a) [2] How many coefficients of the polynomial $(1+x+x^2)^n$ are not divisible by 3?

(b) [3–] How many coefficients of the polynomial $(1+x+x^2)^n$ are odd?

(c) [2+] How many coefficients of the polynomial $\prod_{1\le i<j\le n}(x_i+x_j)$ are odd?

16. [3–]*

(a) Let p be a prime, and let A be the matrix $A = \left[\binom{j+k}{k}\right]_{j,k=0}^{p-1}$, taken over the field $\mathbb{F}_p$. Show that $A^3 = I$, the identity matrix. (Note that A vanishes below the main antidiagonal, i.e., $A_{jk} = 0$ if $j+k \ge p$.)

(b) How many eigenvalues of A are equal to 1?

17. (a) [1+]* Let $m, n \in \mathbb{N}$. Prove the identity $\left(\!\binom{n}{m}\!\right) = \left(\!\binom{m+1}{n-1}\!\right)$.

(b) [2–] Give a combinatorial proof.

18. [2+]* Find a *simple* description of all $n \in \mathbb{P}$ with the following property: there exists $k \in [n]$ such that $\binom{n}{k-1}, \binom{n}{k}, \binom{n}{k+1}$ are in arithmetic progression.

19. (a) [2+] Let $a_1, \ldots, a_n \in \mathbb{N}$. Show that when we expand the product

$$\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-\frac{x_i}{x_j}\right)^{a_i}$$

as a Laurent polynomial in $x_1, \ldots, x_n$ (i.e., negative exponents allowed), then the constant term is the multinomial coefficient $\binom{a_1+\cdots+a_n}{a_1,\ldots,a_n}$.

Hint: First prove the identity

$$1 = \sum_{i=1}^{n}\prod_{j\ne i}\left(1-\frac{x_i}{x_j}\right)^{-1}. \qquad (1.120)$$

(b) [2–] Put $n = 3$ to deduce the identity

$$\sum_{k=-a}^{a} (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \binom{a+b+c}{a,b,c}.$$

(Set $\binom{m}{i} = 0$ if $i < 0$.) Note that if we specialize $a = b = c$, then we obtain

$$\sum_{k=0}^{2a} (-1)^k \binom{2a}{k}^3 = \binom{3a}{a,a,a}.$$

(c) [3+] Let q be an additional indeterminate. Show that when we expand the product

$$\prod_{1\le i<j\le k} \left(1 - q\frac{x_i}{x_j}\right)\left(1 - q^2\frac{x_i}{x_j}\right)\cdots\left(1 - q^{a_i}\frac{x_i}{x_j}\right)$$
$$\cdot\left(1 - \frac{x_j}{x_i}\right)\left(1 - q\frac{x_j}{x_i}\right)\cdots\left(1 - q^{a_j-1}\frac{x_j}{x_i}\right) \quad (1.121)$$

as a Laurent polynomial in $x_1, \dots, x_n$ (whose coefficients are now polynomials in q), then the constant term is the q-multinomial coefficient $\binom{\boldsymbol{a_1+\cdots+a_n}}{\boldsymbol{a_1,\dots,a_n}}$.

(d) [3+] Let $k \in \mathbb{P}$. When the product

$$\prod_{1\le i<j\le n} \left[\left(1 - \frac{x_i}{x_j}\right)\left(1 - \frac{x_j}{x_i}\right)(1 - x_ix_j)\left(1 - \frac{1}{x_ix_j}\right)\right]^k$$

is expanded as above, show that the constant term is

$$\binom{k}{k}\binom{3k}{k}\binom{5k}{k}\cdots\binom{(2n-3)k}{k}\cdot\binom{(n-1)k}{k}.$$

(e) [3–] Let $f(a_1, a_2, \dots, a_n)$ denote the constant term of the Laurent polynomial

$$\prod_{i=1}^{n} \left(q^{-a_i} + q^{-a_i+1} + \cdots + q^{a_i}\right),$$

where each $a_i \in \mathbb{N}$. Show that

$$\sum_{a_1,\dots,a_n\ge 0} f(a_1,\dots,a_n)x_1^{a_1}\cdots x_n^{a_n}$$
$$= (1+x_1)\cdots(1+x_n)\sum_{i=1}^{n} \frac{x_i^{n-1}}{(1-x_i^2)\prod_{j\ne i}(x_i-x_j)(1-x_ix_j)}.$$

20. [2]* How many $m \times n$ matrices of 0's and 1's are there, such that every row and column contains an even number of 1's? An odd number of 1's?

21. [2]* Fix $n \in \mathbb{P}$. In how many ways (as a function of n) can one choose a composition α of n, and then choose a composition of each part of α? (Give an elegant combinatorial proof.)

22. (a) [2] Find the number of compositions of $n > 1$ with an even number of even parts. Naturally a combinatorial proof is preferred.

    (b) [2+] Let $e(n), o(n)$, and $k(n)$ denote, respectively, the number of partitions of n with an even number of even parts, with an odd number of even parts, and that are self-conjugate. Show that $e(n) - o(n) = k(n)$. Is there a simple combinatorial proof?

23. [2] Give a simple "balls into boxes" proof that the total number of parts of all compositions of n is equal to $(n+1)2^{n-2}$. (The simplest argument expresses the answer as a sum of two terms.)

24. [2+] Let $1 \leq k < n$. Give a combinatorial proof that among all 2^{n-1} compositions of n, the part k occurs a total of $(n-k+3)2^{n-k-2}$ times. For instance, if $n = 4$ and $k = 2$, then the part 2 appears once in $2+1+1$, $1+2+1$, $1+1+2$, and twice in $2+2$, for a total of five times.

25. [2+] Let $n - r = 2k$. Show that the number $f(n, r, s)$ of compositions of n with r odd parts and s even parts is given by $\binom{r+s}{r}\binom{r+k-1}{r+s-1}$. Give a generating function proof and a bijective proof.

26. [2]* Let $\bar{c}(m, n)$ denote the number of compositions of n with largest part at most m. Show that
$$\sum_{n \geq 0} \bar{c}(m, n) x^n = \frac{1-x}{1-2x+x^{m+1}}.$$

27. [2+] Find a simple explicit formula for the number of compositions of $2n$ with largest part exactly n.

28. [2]* Let $\kappa(n, j, k)$ be the number of weak compositions of n into k parts, each part less than j. Give a generating function proof that
$$\kappa(n, j, k) = \sum_{r+sj=n} (-1)^s \binom{k+r-1}{r}\binom{k}{s},$$
where the sum is over all pairs $(r, s) \in \mathbb{N}^2$ satisfying $r + sj = n$.

29. [2]* Fix $k, n \in \mathbb{P}$. Show that
$$\sum a_1 \cdots a_k = \binom{n+k-1}{2k-1},$$
where the sum ranges over all compositions $(a_1, \ldots, a_k)$ of n into k parts.

30. [2] Fix $1 \leq k \leq n$. How many integer sequences $1 \leq a_1 < a_2 < \cdots < a_k \leq n$ satisfy $a_i \equiv i \pmod 2$ for all i?

31. [2+]

(a) Let $\#N = n$, $\#X = x$. Find a simple explicit expression for the number of ways of choosing a function $f : N \to X$ and then linearly ordering each block of the coimage of f. (The elements of N and X are assumed to be distinguishable.)

(b) How many ways as in (a) are there if f must be surjective? (Give a simple explicit answer.)

(c) How many ways as in (a) are there if the elements of X are indistinguishable? (Express your answer as a finite sum.)

32. [2] Fix positive integers n and k. Let $\#S = n$. Find the number of k-tuples $(T_1, T_2, \ldots, T_k)$ of subsets T_i of S subject to each of the following conditions *separately*, i.e., the three parts are independent problems (all with the same general method of solution).

(a) $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$

(b) The T_i's are pairwise disjoint.

(c) $T_1 \cup T_2 \cup \cdots \cup T_k = S$

33. (a) [2–]* Let $k, n \geq 1$. Find the number of sequences $\emptyset = S_0, S_1, \ldots, S_k$ of subsets of $[n]$ if for all $1 \leq i \leq k$ we have either (i) $S_{i-1} \subset S_i$ and $|S_i - S_{i-1}| = 1$, or (ii) $S_i \subset S_{i-1}$ and $|S_{i-1} - S_i| = 1$.

(b) [2+]* Suppose that we add the additional condition that $S_k = \emptyset$. Show that now the number $f_k(n)$ of sequences is given by

$$f_k(n) = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (n-2i)^k.$$

Note that $f_k(n) = 0$ if k is odd.

34. [2] Fix $n, j, k \in \mathbb{P}$. How many integer sequences are there of the form $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, where $a_{i+1} - a_i \geq j$ for all $1 \leq i \leq k-1$?

35. The *Fibonacci numbers* are defined by $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ if $n \geq 3$. Express the following numbers in terms of the Fibonacci numbers.

(a) [2–] The number of subsets S of the set $[n] = \{1, 2, \ldots, n\}$ such that S contains no two consecutive integers.

(b) [2] The number of compositions of n into parts greater than 1.

(c) [2–] The number of compositions of n into parts equal to 1 or 2.

(d) [2] The number of compositions of n into odd parts.

(e) [2] The number of sequences $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of 0's and 1's such that $\varepsilon_1 \leq \varepsilon_2 \geq \varepsilon_3 \leq \varepsilon_4 \geq \varepsilon_5 \leq \cdots$.

(f) [2+] $\sum a_1 a_2 \cdots a_k$, where the sum is over all 2^{n-1} compositions $a_1 + a_2 + \cdots + a_k = n$.

(g) [2+] $\sum (2^{a_1-1}-1)\cdots(2^{a_k-1}-1)$, summed over the same set as in (f).

(h) [2+] $\sum 2^{\#\{i\,:\,a_i=1\}}$, summed over the same set as (f).

(i) [2+] $\sum (-1)^{k-1}(5^{a_1-1}+1)\cdots(5^{a_k-1}+1)$, summed over the same set as (f).

(j) [2+]* The number of sequences $(\delta_1,\delta_2,\ldots,\delta_n)$ of 0's, 1's, and 2's such that 0 is never immediately followed by 1.

(k) [2+] The number of distinct terms of the polynomial

$$P_n = \prod_{j=1}^{n}(1+x_j+x_{j+1}).$$

For instance, setting $x_1=a$, $x_2=b$, $x_3=c$, we have $P_2 = 1+a+2b+c+ab+b^2+ac+bc$, which has eight distinct terms.

36. [2] Fix $k,n\in\mathbb{P}$. Find a simple expression involving Fibonacci numbers for the number of sequences $(T_1,T_2,\ldots,T_k)$ of subsets T_i of $[n]$ such that

$$T_1 \subseteq T_2 \supseteq T_3 \subseteq T_4 \supseteq \cdots.$$

37. [2] Show that

$$F_{n+1} = \sum_{k=0}^{n}\binom{n-k}{k}. \tag{1.122}$$

38. [2]* Show that the number of permutations $w\in\mathfrak{S}_n$ fixed by the fundamental transformation $\mathfrak{S}_n \xrightarrow{\wedge} \mathfrak{S}_n$ of Proposition 1.3.1 (i.e., $w=\widehat{w}$) is the Fibonacci number F_{n+1}.

39. [2+] Show that the number of ordered pairs (S,T) of subsets of $[n]$ satisfying $s>\#T$ for all $s\in S$ and $t>\#S$ for all $t\in T$ is equal to the Fibonacci number F_{2n+2}.

40. [2]* Suppose that n points are arranged on a circle. Show that the number of subsets of these points containing no two that are consecutive is the Lucas number L_n. This result shows that the Lucas number L_n may be regarded as a "circular analogue" of the Fibonacci number F_{n+2} (via Exercise 1.35(a)). For further explication, see Example 4.7.16.

41. (a) [2] Let $f(n)$ be the number of ways to choose a subset $S\subseteq[n]$ and a permutation $w\in\mathfrak{S}_n$ such that $w(i)\notin S$ whenever $i\in S$. Show that $f(n)=F_{n+1}n!$.

(b) [2+] Suppose that in (a) we require w to be an n-cycle. Show that the number of ways is now $g(n)=L_n(n-1)!$, where L_n is a Lucas number.

42. [3] Let

$$\begin{aligned} F(x) &= \prod_{n\geq 2}(1-x^{F_n}) = (1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^8)\cdots \\ &= 1-x-x^2+x^4+x^7-x^8+x^{11}-x^{12}-x^{13}+x^{14}+x^{18}+\cdots. \end{aligned}$$

Show that every coefficient of $F(x)$ is equal to $-1, 0$ or 1.

43. [2–] Using only the combinatorial definitions of the Stirling numbers $S(n,k)$ and $c(n,k)$, give formulas for $S(n,1)$, $S(n,2)$, $S(n,n)$, $S(n,n-1)$, $S(n,n-2)$ and $c(n,1)$, $c(n,2)$, $c(n,n)$, $c(n,n-1)$, $c(n,n-2)$. For the case $c(n,2)$, express your answer in terms of the harmonic number $H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$ for suitable m.

44. (a) [2]* Show that the total number of cycles of all even permutations of $[n]$ and the total number of cycles of all odd permutations of $[n]$ differ by $(-1)^n(n-2)!$. Use generating functions.

    (b) [3–]* Give a bijective proof.

45. [2+] Let $S(n,k)$ denote a Stirling number of the second kind. The generating function $\sum_n S(n,k)x^n = x^k/(1-x)(1-2x)\cdots(1-kx)$ implies the identity

$$S(n,k) = \sum 1^{a_1-1}2^{a_2-1}\cdots k^{a_k-1}, \tag{1.123}$$

the sum being over all compositions $a_1 + \cdots + a_k = n$. Give a *combinatorial* proof of (1.123) analogous to the second proof of Proposition 1.3.7. That is, we want to associate with each partition π of $[n]$ into k blocks a composition $a_1 + \cdots + a_k = n$ such that exactly $1^{a_1-1}2^{a_2-1}\cdots k^{a_k-1}$ partitions π are associated with this composition.

46. (a) [2] Let $n, k \in \mathbb{P}$, and let $j = \lfloor k/2 \rfloor$. Let $S(n,k)$ denote a Stirling number of the second kind. Give a generating function proof that

$$S(n,k) \equiv \binom{n-j-1}{n-k} \pmod{2}.$$

    (b) [3–] Give a combinatorial proof.

    (c) [2] State and prove an analogous result for Stirling numbers of the first kind.

47. Let D be the operator $\frac{d}{dx}$.

    (a) [2]* Show that $(xD)^n = \sum_{k=0}^n S(n,k)x^k D^k$.

    (b) [2]* Show that

$$x^n D^n = xD(xD-1)(xD-2)\cdots(xD-n+1) = \sum_{k=0}^n s(n,k)(xD)^k.$$

    (c) [2+]* Find the coefficients $a_{n,i,j}$ in the expansion

$$(x+D)^n = \sum_{i,j} a_{n,i,j} x^i D^j.$$

48. (a) [3] Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_i \geq 0$, be a polynomial all of whose zeros are negative real numbers. Regard $a_k/P(1)$ as the probability of choosing k, so we have a probability distribution on $[0,n]$. Let $\mu = \frac{1}{P(1)}\sum_k k a_k = P'(1)/P(1)$,

the *mean* of the distribution; and let m be the *mode*, i.e., $a_m = \max_k a_k$. Show that

$$|\mu - m| < 1.$$

More precisely, show that

$$\begin{aligned} m = k, & \quad \text{if } k \le \mu < k + \tfrac{1}{k+2} \\ m = k, \text{ or } k+1, \text{ or both}, & \quad \text{if } k + \tfrac{1}{k+2} \le \mu \le k + 1 - \tfrac{1}{n-k+1} \\ m = k+1, & \quad \text{if } k + 1 - \tfrac{1}{n-k+1} < \mu \le k+1. \end{aligned}$$

(b) [2] Fix n. Show that the signless Stirling number $c(n,k)$ is maximized at $k = \lfloor 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \rfloor$ or $k = \lceil 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \rceil$. In particular, $k \sim \log(n)$.

(c) [3] Let $S(n,k)$ denote a Stirling number of the second kind, and define K_n by $S(n, K_n) \ge S(n,k)$ for all k. Let t be the solution of the equation $te^t = n$. Show that for sufficiently large n (and probably all n), either $K_n + 1 = \lfloor e^t \rfloor$ or $K_n + 1 = \lceil e^t \rceil$.

49. (a) [2+] Deduce from equation (1.38) that all the (complex) zeros of $A_d(x)$ are real and simple. (Use Rolle's theorem.)

(b) [2–]* Deduce from Exercise 1.133(b) that the polynomial $\sum_{k=1}^{n} k!\, S(n,k) x^k$ has only real zeros.

50. A sequence $\alpha = (a_0, a_1, \dots, a_n)$ of real numbers is *unimodal* if for some $0 \le j \le n$ we have $a_0 \le a_1 \le \cdots \le a_j \ge a_{j+1} \ge a_{j+2} \ge \cdots \ge a_n$, and is *log-concave* if $a_i^2 \ge a_{i-1}a_{i+1}$ for $1 \le i \le n-1$. We also say that α has *no internal zeros* if there does not exist $i < j < k$ with $a_i \ne 0$, $a_j = 0$, $a_k \ne 0$, and that α is *symmetric* if $a_i = a_{n-i}$ for all i. Define a polynomial $P(x) = \sum a_i x^i$ to be unimodal, log-concave, etc., if the sequence $(a_0, a_1, \dots, a_n)$ of coefficients has that property.

(a) [2–]* Show that a log-concave sequence of nonnegative real numbers with no internal zeros is unimodal.

(b) [2+] Let $P(x) = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} \binom{n}{i} b_i x^i \in \mathbb{R}[x]$. Show that if all the zeros of $P(x)$ are real, then the sequence $(b_0, b_1, \dots, b_n)$ is log-concave. (When all $a_i \ge 0$, this statement is stronger than the assertion that $(a_0, a_1, \dots, a_n)$ is log-concave.)

(c) [2+] Let $P(x) = \sum_{i=0}^{m} a_i x^i$ and $Q(x) = \sum_{i=0}^{n} b_i x^i$ be symmetric, unimodal, and have nonnegative coefficients. Show that the same is true for $P(x)Q(x)$.

(d) [2+] Let $P(x)$ and $Q(x)$ be log-concave with no internal zeros and nonnegative coefficients. Show that the same is true for $P(x)Q(x)$.

(e) [2] Show that the polynomials $\sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)}$ and $\sum_{w \in \mathfrak{S}_n} x^{\mathrm{inv}(w)}$ are symmetric and unimodal.

(f) [4–] Let $1 \le p \le n-1$. Given $w = a_1 \cdots a_n \in \mathfrak{S}_n$, define

$$\mathrm{des}_p(w) = \#\{(i,j) \,:\, i < j \le i + p,\ a_i > a_j\}.$$

Thus $\mathrm{des}_1 = \mathrm{des}$ and $\mathrm{des}_{n-1} = \mathrm{inv}$. Show that the polynomial $\sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}_p(w)}$ is symmetric and unimodal.

(g) [2+] Let S be a subset of $\{(i,j) : 1 \le i < j \le n\}$. An *$S$-inversion* of $w = a_1 \cdots a_n \in \mathfrak{S}_n$ is a pair $(i,j) \in S$ for which $a_i > a_j$. Let $\mathrm{inv}_S(w)$ denote the number of S-inversions of w. Find a set S (for a suitable value of n) for which the polynomial $P_S(x) := \sum_{w \in \mathfrak{S}_n} x^{\mathrm{inv}_S(w)}$ is not unimodal.

51. [3–] Let $k, n \in \mathbb{P}$ with $k \le n$. Let $V(n,k)$ denote the volume of the region $\mathcal{R}_{nk}$ in $\mathbb{R}^n$ defined by

$$0 \le x_i \le 1, \text{ for } 1 \le i \le n$$

$$k - 1 \le x_1 + x_2 + \cdots + x_n \le k.$$

Show that $V(n,k) = A(n,k)/n!$, where $A(n,k)$ is an Eulerian number.

52. [3–] Fix $b \ge 2$. Choose n random N-digit integers in base b (allowing intial digits equal to 0). Add these integers using the usual addition algorithm. For $0 \le j \le n-1$, let $f(j)$ be the number of times that we carry j in the addition process. For instance, if we add 71801, 80914, and 62688 in base 10, then $f(0) = 1$ and $f(1) = f(2) = 2$. Show that as $N \to \infty$, the expected value of $f(j)/N$ (i.e., the expected proportion of the time we carry a j) approaches $A(n, j+1)/n!$, where $A(n,k)$ is an Eulerian number.

53. (a) [2]* The *Eulerian Catalan number* is defined by $EC_n = A(2n+1, n+1)/(n+1)$. The first few Eulerian Catalan numbers, beginning with $EC_0 = 1$, are 1, 2, 22, 604, 31238. Show that $EC_n = 2A(2n, n+1)$, whence $EC_n \in \mathbb{Z}$.

(b) [3–]* Show that EC_n is the number of permutations $w = a_1 a_2 \cdots a_{2n+1}$ with n descents, such that every left factor $a_1 a_2 \cdots a_i$ has at least as many ascents as descents. For $n = 1$ we are counting the two permutations 132 and 231.

54. [2]* How many n-element multisets on $[2m]$ are there satisfying: (i) $1, 2, \ldots, m$ appear at most once each, and (ii) $m+1, m+2, \ldots, 2m$ appear an even number of times each?

55. [2–]* If $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ then let $w^r = a_n \cdots a_2 a_1$, the reverse of w. Express $\mathrm{inv}(w^r)$, $\mathrm{maj}(w^r)$, and $\mathrm{des}(w^r)$ in terms of $\mathrm{inv}(w)$, $\mathrm{maj}(w)$, and $\mathrm{des}(w)$, respectively.

56. [2+] Let M be a finite multiset on $\mathbb{P}$. Generalize equation (1.41) by showing that

$$\sum_{w \in \mathfrak{S}_M} q^{\mathrm{inv}(w)} = \sum_{w \in \mathfrak{S}_M} q^{\mathrm{maj}(w)},$$

where $\mathrm{inv}(w)$ and $\mathrm{maj}(w)$ are defined in Section 1.7. Try to give a proof based on results in Section 1.4 rather than generalizing the proof of (1.41).

57. [2+] Let $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$. Show that the following conditions are equivalent.

(i) Let $C(i)$ be the set of indices j of the columns C_j that intersect the ith row of the diagram $D(w)$ of w. For instance, if $w = 314652$ as in Figure 1.5, then $C(1) = \{1, 2\}$, $C(3) = \{2\}$, $C(4) = \{2, 5\}$, $C(5) = \{2\}$, and all other $C(i) = \emptyset$. Then for every i, j, either $C(i) \subseteq C(j)$ or $C(j) \subseteq C(i)$.

(ii) Let $\lambda(w)$ be the entries of the inversion table $I(w)$ of w written in decreasing order. For instance, $I(52413) = (3,1,2,1,0)$ and $\lambda(52413) = (3,2,1,1,0)$. Regard λ as a partition of $\mathrm{inv}(w)$. Then $\lambda(w^{-1}) = \lambda(w)'$, the conjugate partition to $\lambda(w)$.

(iii) The permutation w is 2143-*avoiding*, i.e., there do not exist $a < b < c < d$ for which $w_b < w_a < w_d < w_c$.

58. For $u \in \mathfrak{S}_k$, let $s_u(n) = \#\mathcal{S}_u(n)$, the number of permutations $w \in \mathfrak{S}_n$ avoiding u. If also $v \in \mathfrak{S}_k$, then write $u \sim v$ if $s_u(n) = s_v(n)$ for all $n \geq 0$ (an obvious equivalence relation). Thus by the discussion preceding Proposition 1.5.1, $u \sim v$ for all $u, v \in \mathfrak{S}_3$.

(a) [2]* Let $u, v \in \mathfrak{S}_k$. Suppose that the permutation matrix P_v can be obtained from P_u by one of the eight dihedral symmetries of the square. For instance, $P_{u^{-1}}$ and be obtained from P_u by reflection in the main diagonal. Show that $u \sim v$. We then say that u and v are *equivalent by symmetry*, denoted $u \approx v$. Thus $\approx$ is a finer equivalence relation than $\sim$. What are the $\approx$ equivalence classes for $\mathfrak{S}_3$?

(b) [3] Show that there are exactly three $\sim$ equivalence classes for $\mathfrak{S}_4$. The equivalence classes are given by $\{1234, 1243, 2143, \dots\}$, $\{3142, 1342, \dots\}$, and $\{1342, \dots\}$, where the omitted permutations are obtained by $\approx$ equivalence.

59. [3] Let $s_u(n)$ have the meaning of the previous exercise. Show that $c_u := \lim_{n\to\infty} s_u(n)^{1/n}$ exists and satisfies $1 < c_u < \infty$.

60. [2+] Define two permutations in $\mathfrak{S}_n$ to be *equivalent* if one can be obtained from the other by interchanging adjacent letters that differ by at least two, an obvious equivalence relation. For instance, when $n = 3$ we have the four equivalence classes $\{123\}$, $\{132, 312\}$, $\{213, 231\}$, $\{321\}$. Describe the equivalence classes in terms of more familiar objects. How many equivalence classes are there?

61. (a) [3–] Let $w = w_1 \cdots w_n$. Let

$$F(x; a, b, c, d) = \sum_{n\geq 1} \sum_{w \in \mathfrak{S}_n} a^{v(w)} b^{p(w)-1} c^{r(w)} d^{f(w)} \frac{x^n}{n!},$$

where $v(w)$ denotes the number of valleys w_i of w for $1 \leq i \leq n$ (where $w_0 = w_{n+1} = 0$ as preceding Proposition 1.5.3), $p(w)$ the number of peaks, $r(w)$ the number of double rises, and $f(w)$ the number of double falls. For instance, if $w = 32451$, then 3 is a peak, 2 is a valley, 4 is a double rise, 5 is a peak, and 1 is a double fall. Thus

$$\begin{aligned} F(x; a, b, c, d) &= x + (c+d)\frac{x^2}{2!} + (c^2 + d^2 + 2ab + 2cd)\frac{x^3}{3!} \\ &\quad + (c^3 + d^3 + 3cd^2 + 3c^2d + 8abc + 8abd)\frac{x^4}{4!} + \cdot. \end{aligned}$$

Show that

$$F(x; a, b, c, d) = \frac{e^{vx} - e^{ux}}{ve^{ux} - ue^{vx}}, \tag{1.124}$$

where $uv = ab$ and $u + v = c + d$. In other words, u and v are zeros of the polynomial $z^2 - (c + d)z + ab$; it makes no difference which zero we call u and which v.

(b) [2–] Let $r(n, k)$ be the number of permutations $w \in \mathfrak{S}_n$ with k peaks. Show that

$$\sum_{n\geq 0}\sum_{k\geq 0} r(n,k)t^k \frac{x^n}{n!} = \frac{1 + u\tan(xu)}{1 - \frac{\tan(xu)}{u}}, \tag{1.125}$$

where $u = \sqrt{t-1}$.

(c) [2+] A *proper double fall* or *proper double descent* of a permutation $w = a_1a_2\cdots a_n$ is an index $1 < i < n$ for which $a_{i-1} > a_i > a_{i+1}$. (Compare with the definition of a double fall or double descent, where we also allow $i = 1$ and $i = n$ with the convention $a_0 = a_{n+1} = 0$.) Let $f(n)$ be the number of permutations $w \in \mathfrak{S}_n$ with no proper double descents. Show that

$$\begin{aligned} \sum_{n\geq 0} f(n)\frac{x^n}{n!} &= \frac{1}{\displaystyle\sum_{j\geq 0}\left(\frac{x^{3j}}{(3j)!} - \frac{x^{3j+1}}{(3j+1)!}\right)} \qquad (1.126)\\ &= 1 + x + 2\frac{x^2}{2!} + 5\frac{x^3}{3!} + 17\frac{x^4}{4!} + 70\frac{x^5}{5!} + 349\frac{x^6}{6!}\\ &\quad +2017\frac{x^7}{7!} + 13358\frac{x^8}{8!} + \cdots. \end{aligned}$$

62. In this exercise we consider one method for generalizing the disjoint cycle decomposition of permutations of sets to multisets. A *multiset cycle* of $\mathbb{P}$ is a sequence $C = (i_1, i_2, \ldots, i_k)$ of positive integers with repetitions allowed, where we regard $(i_1, i_2, \ldots, i_k)$ as equivalent to $(i_j, i_{j+1}, \ldots, i_k, i_1, \ldots, i_{j-1})$ for $1 \leq j \leq k$. Introduce indeterminates $x_1, x_2, \ldots$, and define the *weight* of C by $w(C) = x_{i_1}\cdots x_{i_k}$. A *multiset permutation* or *multipermutation* of a multiset M is a multiset of multiset cycles, such that M is the multiset of all elements of the cycles. For instance, the multiset $\{1, 1, 2\}$ has the following four multipermutations: (1)(1)(2), (11)(2), (12)(1), (112). The *weight* $w(\pi)$ of a multipermutation $\pi = C_1C_2\cdots C_j$ is given by $w(\pi) = w(C_1)\cdots w(C_j)$.

(a) [2–]* Show that

$$\prod_C (1 - w(C))^{-1} = \sum_\pi w(\pi),$$

where C ranges over all multiset cycles on $\mathbb{P}$ and π over all (finite) multiset permutations on $\mathbb{P}$.

(b) [2+] Let $p_k = x_1^k + x_2^k + \cdots$. Show that

$$\prod_C (1 - w(C))^{-1} = \prod_{k\geq 1}(1 - p_k)^{-1}.$$

(c) [1+] Let $f_k(n)$ denote the number of multiset permutations on $[k]$ of total size n. For instance, $f_2(3) = 14$, given by (1)(1)(1), (1)(1)(2), (1)(2)(2), (2)(2)(2), (11)(1), (11)(2), (12)(1), (12)(2), (22)(1), (22)(2), (111), (112), (122), (222). Deduce from (b) that

$$\sum_{n\geq 0} f_k(n)x^n = \prod_{i\geq 1}(1-kx^i)^{-1}.$$

(d) [3–] Find a direct combinatorial proof of (b) or (c).

63. (a) [2–] We are given n square envelopes of different sizes. In how many different ways can they be arranged by inclusion? For instance, if $n = 3$ there are six ways; namely, label the envelopes A, B, C with A the largest and C the smallest, and let $I \in J$ mean that envelope I is contained in envelope J. Then the six ways are: (1) $\emptyset$, (2) $B \in A$, (3) $C \in A$, (4) $C \in B$, (5) $B \in A$, $C \in A$, (6) $C \in B \in A$.

(b) [2] How many arrangements have exactly k envelopes that are not contained in another envelope? That don't contain another envelope?

64. (a) [2] Let $f(n)$ be the number of sequences $a_1, \dots, a_n$ of positive integers such that for each $k > 1$, k only occurs if $k-1$ occurs before the last occurrence of k. Show that $f(n) = n!$. (For $n = 3$ the sequences are 111, 112, 121, 122, 212, 123.)

(b) [2] Show that $A(n, k)$ of these sequences satisfy $\max\{a_1, \dots, a_n\} = k$.

65. [3] Let $y = \prod_{n\geq 1}(1-x^n)^{-1}$. Show that

$$4y^3y'' + 5xy^3y''' + x^2y^3y^{(\mathrm{iv})} - 16y^2y'^2 - 15xy^2y'y'' + 20x^2y^2y'y'''$$
$$-19x^2y^2y''^2 + 10xyy'^3 + 12x^2yy'^2y'' + 6x^2y'^4 = 0. \qquad (1.127)$$

66. [2–]* Let $p_k(n)$ denote the number of partitions of n into k parts. Give a bijective proof that

$$p_0(n) + p_1(n) + \cdots + p_k(n) = p_k(n+k).$$

67. [2–]* Express the number of partitions of n with no part equal to 1 in terms of values $p(k)$ of the partition function.

68. [2]* Let $n \geq 1$, and let $f(n)$ be the number of partitions of n such that for all k, the part k occurs at most k times. Let $g(n)$ be the number of partitions of n such that no part has the form $i(i+1)$, i.e., no parts equal to $2, 6, 12, 20, \dots$. Show that $f(n) = g(n)$.

69. [2]* Let $f(n)$ denote the number of self-conjugate partitions of n all of whose parts are even. Express the generating function $\sum_{n\geq 0} f(n)x^n$ as a simple product.

70. (a) [2] Find a bijection between partitions $\lambda \vdash n$ of rank r and integer arrays

$$A_\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

such that $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$, and $r + \sum(a_i + b_i) = n$.

(b) [2+] A *concatenated spiral self-avoiding walk* (CSSAW) on the square lattice is a lattice path in the plane starting at $(0,0)$, with steps $(\pm 1, 0)$ and $(0, \pm 1)$ and first step $(1,0)$, with the following three properties: (i) the path is *self-avoiding*, i.e, it never returns to a previously visited lattice point, (ii) every step after the first must continue in the direction of the previous step or turn right, and (iii) at the end of the walk it must be possible to turn right and walk infinitely many steps in the direction faced without intersecting an earlier part of the path. For instance, writing $N = (0,1)$, etc., the five CSSAW's of length four are $NNNN$, $NNNE$, $NNEE$, $NEEE$, and $NESS$. Note for instance that $NEES$ is not a CSSAW since continuing with steps $WWW\cdots$ will intersect $(0,0)$. Show that the number of CSSAW's of length n is equal to $p(n)$, the number of partitions of n.

71. [2+] How many pairs (λ, μ) of partitions of integers are there such that $\lambda \vdash n$, and the Young diagram of μ is obtained from the Young diagram of λ by adding a single square? Express your answer in terms of the partition function values $p(k)$ for $k \leq n$. Give a simple combinatorial proof.

72. (a) [3–] Let $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ be partitions. Define $\mu \leq \lambda$ if $\mu_i \leq \lambda_i$ for all i. Show that

$$\sum_{\mu \leq \lambda} q^{|\mu|+|\lambda|} = \frac{1}{(1-q)(1-q^2)^2(1-q^3)^2(1-q^4)^2\cdots}. \tag{1.128}$$

(b) [3–] Show that the number of pairs (λ, μ) such that λ and μ have *distinct* parts, $\mu \leq \lambda$ as in (a), and $|\lambda| + |\mu| = n$, is equal to $p(n)$, the number of partitions of n. For instance, when $n = 5$ we have the seven pairs $(\emptyset, 5)$, $(\emptyset, 41)$, $(\emptyset, 32)$, $(1,4)$, $(2,3)$, $(1,31)$, and $(2,21)$.

73. [2] Let λ be a partition. Show that

$$\begin{aligned}
\sum_i \left\lceil \frac{\lambda_{2i-1}}{2} \right\rceil &= \sum_i \left\lceil \frac{\lambda'_{2i-1}}{2} \right\rceil \\
\sum_i \left\lfloor \frac{\lambda_{2i-1}}{2} \right\rfloor &= \sum_i \left\lceil \frac{\lambda'_{2i}}{2} \right\rceil \\
\sum_i \left\lfloor \frac{\lambda_{2i}}{2} \right\rfloor &= \sum_i \left\lfloor \frac{\lambda'_{2i}}{2} \right\rfloor.
\end{aligned}$$

74. [2] Let $p_k(n)$ denote the number of partitions of n into k parts. Fix $t \geq 0$. Show that as $n \to \infty$, $p_{n-t}(n)$ becomes eventually constant. What is this constant $f(t)$? What is the least value of n for which $p_{n-t}(n) = f(t)$? Your arguments should be combinatorial.

75. [2–] Let $p_k(n)$ be as above, and let $q_k(n)$ be the number of partitions of n into k distinct parts. For example, $q_3(8) = 2$, corresponding to $(5,2,1)$ and $(4,3,1)$. Give a simple combinatorial proof that $q_k\left(n + \binom{k}{2}\right) = p_k(n)$.

76. [2] Prove the partition identity

$$\prod_{i\geq 1}(1+qx^{2i-1}) = \sum_{k\geq 0} \frac{x^{k^2}q^k}{(1-x^2)(1-x^4)\cdots(1-x^{2k})}. \tag{1.129}$$

77. [3–] Give a "subtraction-free" bijective proof of the pentagonal number formula by proving directly the identity

$$1+\frac{\sum_{n \text{ odd}}\left(x^{n(3n-1)/2}+x^{n(3n+1)/2}\right)}{\prod_{j\geq 1}(1-x^j)} = \frac{1+\sum_{n \text{ even}}\left(x^{n(3n-1)/2}+x^{n(3n+1)/2}\right)}{\prod_{j\geq 1}(1-x^j)}.$$

78. (a) [2] The *logarithmic derivative* of a power series $F(x)$ is $\frac{d}{dx}\log F(x) = F'(x)/F(x)$. By logarithmically differentiating the power series $\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1}(1-x^i)^{-1}$, derive the recurrence

$$n\cdot p(n) = \sum_{i=1}^{n} \sigma(i)p(n-i),$$

where $\sigma(i)$ is the sum of the divisors of i.

(b) [2+] Give a combinatorial proof.

79. (a) [2+] Given a set $S\subseteq \mathbb{P}$, let $p_S(n)$ (resp. $q_S(n)$) denote the number of partitions of n (resp. number of partitions of n into distinct parts) whose parts belong to S. (These are special cases of the function $p(\mathcal{S}, n)$ of Corollary 1.8.2.) Call a pair (S, T), where $S, T\subseteq \mathbb{P}$, an *Euler pair* if $p_S(n) = q_T(n)$ for all $n\in\mathbb{N}$. Show that (S, T) is an Euler pair if and only if $2T\subseteq T$ (where $2T = \{2i : i\in T\}$) and $S = T - 2T$.

(b) [1+] What is the significance of the case $S = \{1\}$, $T = \{1, 2, 4, 8, \ldots\}$?

80. [2+] If λ is a partition of an integer n, let $f_k(\lambda)$ be the number of times k appears as a part of λ, and let $g_k(\lambda)$ be the number of distinct parts of λ that occur at least k times. For example, $f_2(4,2,2,2,1,1) = 3$ and $g_2(4,2,2,2,1,1) = 2$. Show that $\sum f_k(\lambda) = \sum g_k(\lambda)$, where $k\in\mathbb{P}$ is fixed and both sums range over all partitions λ of a fixed integer $n\in\mathbb{P}$.

81. [2+] A *perfect partition* of $n\geq 1$ is a partition $\lambda\vdash n$ which "contains" precisely one partition of each positive integer $m\leq n$. In other words, regarding λ as the multiset of its parts, for each $m\leq n$ there is a unique submultiset of λ whose parts sum to m. Show that the number of perfect partitions of n is equal to the number of *ordered* factorizations (with any number of factors) of $n+1$ into integers ≥ 2.

Example. The perfect partitions of 5 are $(1,1,1,1,1)$, $(3,1,1)$, and $(2,2,1)$. The ordered factorizations of 6 are $6 = 2\cdot 3 = 3\cdot 2$.

82. [3] Show that the number of partitions of $5n+4$ is divisible by 5.

83. [3–] Let $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$. Define

$$\begin{aligned}\alpha(\lambda) &= \sum_i \lceil \lambda_{2i-1}/2 \rceil \\ \beta(\lambda) &= \sum_i \lfloor \lambda_{2i-1}/2 \rfloor \\ \gamma(\lambda) &= \sum_i \lceil \lambda_{2i}/2 \rceil \\ \delta(\lambda) &= \sum_i \lfloor \lambda_{2i}/2 \rfloor.\end{aligned}$$

Let a, b, c, d be (commuting) indeterminates, and define

$$w(\lambda) = a^{\alpha(\lambda)} b^{\beta(\lambda)} c^{\gamma(\lambda)} d^{\delta(\lambda)}.$$

For instance, if $\lambda = (5,4,4,3,2)$ then $w(\lambda)$ is the product of the entries of the diagram

$$\begin{array}{ccccc} a & b & a & b & a \\ c & d & c & d & \\ a & b & a & b & \\ c & d & c & & \\ a & b & & & \end{array}$$

Show that

$$\sum_{\lambda \in \mathrm{Par}} w(\lambda) = \prod_{j \geq 1} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^j c^{j-1} d^{j-1})(1 - a^j b^{j-1} c^j d^{j-1})}, \tag{1.130}$$

where Par denotes the set of all partitions λ of all integers $n \geq 0$.

84. [2]* Show that the number of partitions of n in which each part appears exactly 2, 3, or 5 times is equal to the number of partitions of n into parts congruent to ± 2, ± 3, $6 \pmod{12}$.

85. [2+]* Prove that the number of partitions of n in which no part appears exactly once equals the number of partitions of n into parts not congruent to $\pm 1 \pmod 6$.

86. [3] Prove that the number of partitions of n into parts congruent to 1 or $5 \pmod 6$ equals the number of partitions of n in which the difference between all parts is at least 3 and between multiples of 3 is at least 6.

87. [3–]* Let $A_k(n)$ be the number of partitions of n into odd parts (repetition allowed) such that exactly k distinct parts occur. For instance, when $n = 35$ and $k = 3$, one of the partitions being enumerated is $(9,9,5,3,3,3,3)$. Let $B_k(n)$ be the number of partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ of n such that the sequence $\lambda_1, \dots, \lambda_r$ is composed of exactly k noncontiguous sequences of one or more consecutive integers. For instance, when $n = 44$ and $k = 3$, one of the partitions being enumerated is $(10,9,8,7,5,3,2)$, which is composed of $10,9,8,7$ and 5 and $3,2$. Show that $A_k(n) = B_k(n)$ for all k and n. Note that summing over all k gives Proposition 1.8.5, i.e., $p_{\mathrm{odd}}(n) = q(n)$.

88. (a) [3] Prove the identities

$$\sum_{n\geq 0} \frac{x^{n^2}}{(1-x)(1-x^2)\cdots(1-x^n)} = \frac{1}{\prod_{k\geq 0}(1-x^{5k+1})(1-x^{5k+4})}$$

$$\sum_{n\geq 0} \frac{x^{n(n+1)}}{(1-x)(1-x^2)\cdots(1-x^n)} = \frac{1}{\prod_{k\geq 0}(1-x^{5k+2})(1-x^{5k+3})}.$$

(b) [2] Show that the identities in (a) are equivalent to the following combinatorial statements:

- The number of partitions of n into parts $\equiv \pm 1 \pmod 5$ is equal to the number of partitions of n whose parts differ by at least 2.
- The number of partitions of n into parts $\equiv \pm 2 \pmod 5$ is equal to the number of partitions of n whose parts differ by at least 2 and for which 1 is not a part.

(c) [2]* Let $f(n)$ be the number of partitions $\lambda \vdash n$ satisfying $\ell(\lambda) = \mathrm{rank}(\lambda)$. Show that $f(n)$ is equal to the number of partitions of n whose parts differ by at least 2.

89. [3] A *lecture hall partition* of length k is a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ (some of whose parts may be 0) satisfying

$$0 \leq \frac{\lambda_k}{1} \leq \frac{\lambda_{k-1}}{2} \leq \cdots \leq \frac{\lambda_1}{k}.$$

Show that the number of lecture hall partitions of n of length k is equal to the number of partitions of n whose parts come from the set $1, 3, 5, \ldots, 2k-1$ (with repetitions allowed).

90. [3] Let $f(n)$ be the number of partitions of n all of whose parts are Lucas numbers L_{2n+1} of odd index. For instance, $f(12) = 5$, corresponding to

$$\begin{array}{l}
1+1+1+1+1+1+1+1+1+1+1+1\\
4+1+1+1+1+1+1+1+1\\
4+4+1+1+1+1\\
4+4+4\\
11+1
\end{array}$$

Let $g(n)$ be the number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_i/\lambda_{i+1} > \frac{1}{2}(3+\sqrt{5})$ whenever $\lambda_{i+1} > 0$. For instance, $g(12) = 5$, corresponding to

$$12, \quad 11+1, \quad 10+2, \quad 9+3, \quad 8+3+1.$$

Show that $f(n) = g(n)$ for all $n \geq 1$.

91. (a) [3–] Show that

$$\sum_{n\in\mathbb{Z}} x^n q^{n^2} = \prod_{k\geq 1}(1-q^{2k})(1+xq^{2k-1})(1+x^{-1}q^{2k-1}).$$

(b) [2] Deduce from (a) the Pentagonal Number Formula (Proposition 1.8.7).

(c) [2] Deduce from (a) the two identities

$$\prod_{k\geq 1}\frac{1-q^k}{1+q^k} = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} \tag{1.131}$$

$$\prod_{k\geq 1}\frac{1-q^{2k}}{1-q^{2k-1}} = \sum_{n\geq 0} q^{\binom{n+1}{2}}. \tag{1.132}$$

(d) [2+] Deduce from (a) the identity

$$\prod_{k\geq 1}(1-q^k)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)/2}.$$

Hint. First substitute $-xq^{-1/2}$ for x and $q^{1/2}$ for q.

92. [3] Let $\mathcal{S}\subseteq\mathbb{P}$ and let $p(\mathcal{S},n)$ denote the number of partitions of n whose parts belong to $\mathcal{S}$. Let

$$\begin{aligned}\mathcal{S} &= \{n : n \text{ odd or } n\equiv \pm 4,\pm 6,\pm 8,\pm 10 \pmod{32}\}\\ \mathcal{T} &= \{n : n \text{ odd or } n\equiv \pm 2,\pm 8,\pm 12,\pm 14 \pmod{32}\}.\end{aligned}$$

Show that $p(\mathcal{S},n)=p(\mathcal{T},n-1)$ for all $n\geq 1$. Equivalently, we have the remarkable identity

$$\prod_{n\in\mathcal{S}}\frac{1}{1-x^n} = 1 + x\prod_{n\in\mathcal{T}}\frac{1}{1-x^n}. \tag{1.133}$$

93. [3] Let

$$\begin{aligned}S &= \pm\{1,4,5,6,7,9,11,13,16,21,23,28 \pmod{66}\}\\ T &= \pm\{1,4,5,6,7,9,11,14,16,17,27,29 \pmod{66}\},\end{aligned}$$

where

$$\pm\{a,b,\ldots \pmod m\} := \{n\in\mathbb{P} : n\equiv \pm a,\pm b,\ldots \pmod m\}.$$

Show that $p(S,n)=p(T,n)$ for all $n\geq 1$ *except* $n=13$. Equivalently, we have another remarkable identity similar to equation (1.133):

$$\prod_{n\in\mathcal{S}}\frac{1}{1-x^n} = x^{13} + \prod_{n\in\mathcal{T}}\frac{1}{1-x^n}.$$

94. (a) [3–] Let $n\geq 0$. Show that the following numbers are equal.

- The number of solutions to $n = \sum_{i \geq 0} a_i 2^i$, where $a_i = 0, 1$, or 2.
- Then number of odd integers k for which the Stirling number $S(n+1, k)$ is odd.
- The number of odd binomial coefficients of the form $\binom{n-k}{k}$, $0 \leq k \leq n$.
- The number of ways to write b_n as a sum of distinct Fibonacci numbers F_n, where

$$\prod_{i \geq 0}(1 + x^{F_{2i}}) = \sum_{n \geq 0} x^{b_n}, \quad b_0 < b_1 < \cdots .$$

(b) [2–] Denote by a_{n+1} the number being counted by (a), so $(a_1, a_2, \ldots, a_{10}) = (1, 1, 2, 1, 3, 2, 3, 1, 4, 3)$. Deduce from (a) that

$$\sum_{n \geq 0} a_{n+1} x^n = \prod_{i \geq 0}(1 + x^{2^i} + x^{2^{i+1}}).$$

(c) [2] Deduce from (a) that $a_{2n} = a_n$ and $a_{2n+1} = a_n + a_{n+1}$.

(d) [3–] Show that every positive rational number can be written in exactly one way as a fraction a_n / a_{n+1}.

95. [3] At time $n = 1$ place a line segment (toothpick) of length one on the xy-plane, centered at $(0, 0)$ and parallel to the y-axis. At time $n > 1$, place additional line segments that are centered at the end and perpendicular to an exposed toothpick end, where an *exposed end* is the end of a toothpick that is neither the end nor the midpoint of another toothpick. Figure 1.28 shows the configurations obtained for times $n \leq 6$. Let $f(n)$ be the total number of toothpicks that have been placed up to time n, and let

$$F(x) = \sum_{n \geq 1} f(n) x^n.$$

Figure 1.28 shows that

$$F(x) = x + 3x^2 + 7x^3 + 11x^4 + 15x^5 + 23x^6 + \cdots .$$

Show that

$$F(x) = \frac{x}{(1-x)(1-2x)} \left(1 + 2x \prod_{k \geq 0} \left(1 + x^{2^k - 1} + 2x^{2^k}\right)\right).$$

96. Define

$$\begin{aligned} x \prod_{n \geq 1}(1 - x^n)^{24} &= \sum_{n \geq 1} \tau(n) x^n \\ &= x - 24x^2 + 252x^3 - 1472x^4 + 4830x^5 - 6048x^6 - 16744x^7 + \cdots . \end{aligned}$$

(a) [3+] Show that $\tau(mn) = \tau(m)\tau(n)$ if m and n are relatively prime.

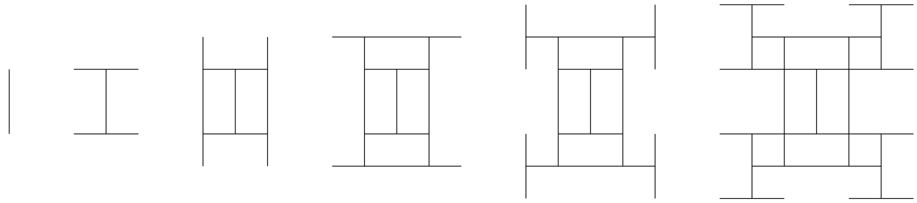


Figure 1.28: The growth of toothpicks

(b) [3+] Show that if p is prime and $n \geq 1$ then

$$\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}).$$

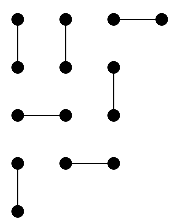
(c) [4] Show that if p is prime then $|\tau(p)| < 2p^{11/2}$. Equivalently, write

$$\sum_{n\geq 0} \tau(p^n)x^n = \frac{P_p(x)}{1-\tau(p)x+p^{11}x^2},$$

so by (b) and Theorem 4.4.1.1 the numerator $P_p(x)$ is a polynomial. Then the zeros of the denominator are not real.

(d) [5] Show that $\tau(n) \neq 0$ for all $n \geq 1$.

97. [3–] Let $f(n)$ be the number of partitions of $2n$ whose Ferrers diagram can be covered by n edges, each connecting two adjacent dots. For instance, $(4,3,3,3,1)$ can be covered as follows:



Show that $\sum_{n\geq 0} f(n)x^n = \prod_{i\geq 1}(1-x^i)^{-2}$.

98. [2+] Let $n, a, k \in \mathbb{N}$ and $\zeta = e^{2\pi i/n}$. Show that

$$\binom{\boldsymbol{na}}{\boldsymbol{k}}_{q=\zeta} = \begin{cases} \binom{a}{b}, & k = nb \\ 0, & \text{otherwise.} \end{cases}$$

99. [2] Let $0 \leq k \leq n$ and $f(q) = \binom{\boldsymbol{n}}{\boldsymbol{k}}$. Compute $f'(1)$. Try to avoid a lot of computation.

100. [2+] State and prove a q-analogue of the Chu-Vandermonde identity

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$$

(Example 1.1.17).

101. [2]* Explain why we cannot set $q = 1$ on both sides of equation (1.85) to obtain the identity

$$1 = \sum_{k \geq 0} \frac{x^k}{k!}.$$

102. (a) [2]* Let x and y be variables satisfying the commutation relation $yx = qxy$, where q commutes with x and y. Show that

$$(x+y)^n = \sum_{k=0}^{n} \binom{\boldsymbol{n}}{\boldsymbol{k}} x^k y^{n-k}.$$

(b) [2]* Generalize to $(x_1 + x_2 + \cdots + x_m)^n$, where $x_i x_j = q x_j x_i$ for $i > j$.

(c) [2+]* Generalize further to $(x_1 + x_2 + \cdots + x_m)^n$, where $x_i x_j = q_j x_j x_i$ for $i > j$, and where the q_j's are variables commuting with all the x_i's and with each other.

103. (a) [3+] Given a partition λ (identified with its Young diagram) and $u \in \lambda$, let $a(u)$ (called the *arm length* of u) denote the number of squares directly to the right of u, counting u itself exactly once. Similarly let $l(u)$ (called the *leg length* of u) denote the number of squares directly below u, counting u itself once. Thus if $u = (i,j)$ then $a(u) = \lambda_i - j + 1$ and $l(u) = \lambda'_j - i + 1$. Define

$$\gamma(\lambda) = \#\{u \in \lambda \,:\, a(u) - l(u) = 0 \text{ or } 1\}.$$

Show that

$$\sum_{\lambda \vdash n} q^{\gamma(\lambda)} = \sum_{\lambda \vdash n} q^{\ell(\lambda)}, \tag{1.134}$$

where $\ell(\lambda)$ denotes the length (number of parts) of λ.

(b) [2]* Clearly the coefficient of x^n in the right-hand side of equation (1.134) is 1. Show directly (without using (a)) that the same is true for the left-hand side.

104. [2+] Let $n \geq 1$. Find the number $f(n)$ of integer sequences $(a_1, a_2, \ldots, a_n)$ such that $0 \leq a_i \leq 9$ and $a_1 + a_2 + \cdots + a_n \equiv 0 \pmod 4$. Give a simple explicit formula (no sums) that depends on the congruence class of n modulo 4.

105. (a) [3–] Let $n \in \mathbb{P}$, and let $f(n)$ denote the number of subsets of $\mathbb{Z}/n\mathbb{Z}$ (the integers modulo n) whose elements sum to 0 in $\mathbb{Z}/n\mathbb{Z}$. For instance, $f(4) = 4$, corresponding to $\emptyset$, $\{0\}$, $\{1,3\}$, $\{0,1,3\}$. Show that

$$f(n) = \frac{1}{n} \sum_{\substack{d \mid n \\ d \text{ odd}}} \phi(d) 2^{n/d},$$

where ϕ denotes Euler's totient function.

(b) [5–] When n is odd, it can be shown using (a) (see Exercise 7.112) that $f(n)$ is equal to the number of necklaces (up to cyclic rotation) with n beads, each bead colored black or white. Give a combinatorial proof. (This is easy if n is prime.)

(c) [5–] Generalize. For instance, investigate the number of subsets S of $\mathbb{Z}/n\mathbb{Z}$ satisfying $\sum_{i\in S} p(i) \equiv \alpha \pmod{n}$, where p is a fixed polynomial and $\alpha \in \mathbb{Z}/n\mathbb{Z}$ is fixed.

106. [2] Let $f(n,k)$ be the number of sequences $a_1a_2\cdots a_n$ of positive integers such that the largest number occurring is k and such that the first occurrence of i appears before the first occurrence of $i+1$ ($1 \leq i \leq k-1$). Express $f(n,k)$ in terms of familiar numbers. Give a combinatorial proof. (It is assumed that every number $1,2,\ldots,k$ occurs at least once.)

107. [1+]* Give a direct combinatorial proof of equation (1.94e), viz.,

$$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i), \quad n \geq 0.$$

108. (a) [2+] Give a combinatorial proof that the number of partitions of $[n]$ such that no two consecutive integers appear in the same block is the Bell number $B(n-1)$.

(b) [2+]* Give a combinatorial proof that the number of partitions of $[n]$ such that no two *cyclically consecutive* integers (i.e., two integers i, j for which $j \equiv i + 1 \pmod{n}$) appear in the same block is equal to the number of partitions of $[n]$ with no singleton blocks.

109. [2+]

(a) Show that the number of permutations $a_1 \cdots a_n \in \mathfrak{S}_n$ for which there is no $1 \leq i < j \leq n-1$ satisfying $a_i < a_j < a_{j+1}$ is equal to the Bell number $B(n)$.

(b) Show that the same conclusion holds if the condition $a_i < a_j < a_{j+1}$ is replaced with $a_i < a_{j+1} < a_j$.

(c) Show that the number of permutations $w \in \mathfrak{S}_n$ satisfying the conditions of *both* (a) and (b) is equal to the number of involutions in $\mathfrak{S}_n$.

110. [3–] Let $f(n)$ be the number of partitions π of $[n]$ such that the union of no proper subset of the blocks of π is an interval $[a,b]$. For instance, $f(4) = 2$, corresponding to the partitions 13-24 and 1234, while $f(5) = 6$. Set $f(0) = 1$. Let

$$F(x) = \sum_{n\geq 0} f(n)x^n = 1 + x + x^2 + x^3 + 2x^4 + 6x^5 + \cdots.$$

Find the coefficients of $(x/F(x))^{\langle -1\rangle}$.

111. [3–] Let $f(n)$ be the number of partitions π of $[n]$ such that no block of π is an interval $[a,b]$ (allowing $a = b$). Thus $f(1) = f(2) = f(3) = 0$ and $f(4) = 1$, corresponding to the partition 13-24. Let

$$F(x) = \sum_{n\geq 0} f(n)x^n = 1 + x^4 + 5x^5 + 21x^6 + \cdots.$$

Express $F(x)$ in terms of the *ordinary* generating function $G(x) = \sum_{n\geq 0} B(n)x^n = 1 + x + 2x^2 + 5x^3 + 15x^4 + \cdots$.

112. [2]* How many permutations $w \in \mathfrak{S}_n$ have the same number of cycles as weak excedances?

113. [2–]* Fix $k, n \in \mathbb{P}$. How many sequences $(T_1, \dots, T_k)$ of subsets T_i of $[n]$ are there such that the *nonempty* T_i form a partition of $[n]$?

114. (a) [2–]* How many permutations $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ have the property that for all $1 \leq i < n$, the numbers appearing in w between i and $i+1$ (whether i is to the left or right of $i+1$) are all less than i? An example of such a permutation is 976412358.

(b) [2–]* How many permutations $a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ satisfy the following property: if $2 \leq j \leq n$, then $|a_i - a_j| = 1$ for some $1 \leq i < j$? Equivalently, for all $1 \leq i \leq n$, the set $\{a_1, a_2, \dots, a_i\}$ consists of consecutive integers (in some order). E.g., for $n = 3$ there are the four permutations 123, 213, 231, 321. More generally, find the number of such permutations with descent set $S \subseteq [n-1]$.

115. [3–] Let $n = 2^{17} + 2$ and define $Q_n(t) = \sum_{S \subseteq [n-1]} t^{\beta_n(S)}$. Show that $e^{2\pi i/n}$ is (at least) a double root of $Q_n(t)$.

116. (a) [2]* Show that the expected number of cycles of a random permutation $w \in \mathfrak{S}_n$ (chosen from the uniform distribution) is given by the harmonic number $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \sim \log n$.

(b) [3] Let $f(n)$ be the expected length of the longest cycle of a random permutation $w \in \mathfrak{S}_n$ (again from the uniform distribibuiton). Show that

$$\lim_{n \to \infty} \frac{f(n)}{n} = \int_0^\infty \exp\left(-x - \int_x^\infty \frac{e^{-y}}{y} dy\right) dx = 0.62432965 \cdots .$$

117. [2+] Let w be a random permutation of $1, 2, \dots, n$ (chosen from the uniform distribution). Fix a positive integer $1 \leq k \leq n$. What is the probability p_{nk} that in the disjoint cycle decomposition of w, the length of the cycle containing 1 is k? In other words, what is the probability that k is the least positive integer for which $w^k(1) = 1$? Give a simple proof avoiding generating functions, induction, etc.

118. (a) [2]* Let w be a random permutation of $1, 2, \dots, n$ (chosen from the uniform distribution), $n \geq 2$. Show that the probability that 1 and 2 are in the same cycle of w is $1/2$.

(b) [2+] Generalize (a) as follows. Let $2 \leq k \leq n$, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \vdash k$, where $\lambda_\ell > 0$. Choose a random permutation $w \in \mathfrak{S}_n$. Let P_λ be the probability that $1, 2, \dots, \lambda_1$ are in the same cycle C_1 of w, and $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$ are in the same cycle C_2 of w different from C_1, etc. Show that

$$P_\lambda = \frac{(\lambda_1 - 1)! \cdots (\lambda_\ell - 1)!}{k!}.$$

(c) [3–] Same as (b), except now we take w uniformly from the alternating group $\mathfrak{A}_n$. Let the resulting probability be Q_λ. Show that

$$Q_\lambda = \frac{(\lambda_1-1)!\cdots(\lambda_\ell-1)!}{(k-2)!}\left(\frac{1}{k(k-1)} + (-1)^{n-\ell}\frac{1}{n(n-1)}\right).$$

119. [2+] Let P_n denote the probability that a random permutation (chosen from the uniform distribution) in $\mathfrak{S}_{2n}$ has all cycle lengths at most n. Show that $\lim_{n\to\infty} P_n = 1 - \log 2 = 0.306852819\cdots$.

120. [2+] Let $E_k(n)$ denote the expected number of k-cycles of a permutation $w \in \mathfrak{S}_n$, as discussed in Example 1.3.5. Give a simple combinatorial explanation of the formula $E_k(n) = 1/k$, $n \geq k$.

121. (a) [2]* Let $f(n)$ denote the number of fixed-point free involutions $w \in \mathfrak{S}_{2n}$ (i.e., $w^2 = 1$, and $w(i) \neq i$ for all $i \in [2n]$). Find a simple expression for $\sum_{n\geq 0} f(n)x^n/n!$. (Set $f(0) = 1$.)

(b) [2–]* If $X \subseteq \mathbb{P}$, then write $-X = \{-i : i \in X\}$. Let $g(n)$ be the number of ways to choose a subset X of $[n]$, and then choose fixed point free involutions w on $X \cup (-X)$ and $\bar{w}$ on $\bar{X} \cup (-\bar{X})$, where $\bar{X} = \{i \in [n] : i \notin X\}$. Use (a) to find a simple expression for $g(n)$.

(c) [2+]* Find a combinatorial proof for the formula obtained for $g(n)$ in (b).

122. [2–]* Find $\sum_w x^{\mathrm{exc}(w)}$, where w ranges over all fixed-point free involutions in $\mathfrak{S}_{2n}$ and $\mathrm{exc}(w)$ denotes the number of excedances of w.

123. [2]* Let $\mathfrak{A}_n$ denote the alternating group on $[n]$, i.e., the group of all permutations with an even number of cycles of even length. Define the *augmented cycle indicator* $\tilde{Z}_{\mathfrak{A}_n}$ of $\mathfrak{A}_n$ by

$$\tilde{Z}_{\mathfrak{A}_n} = \sum_{w\in\mathfrak{A}_n} t^{\mathrm{type}(w)},$$

as in equation (1.25). Show that

$$\sum_{n\geq 0} \tilde{Z}_{\mathfrak{A}_n}\frac{x^n}{n!} = \exp\left(t_1 x + t_3\frac{x^3}{3} + t_5\frac{x^5}{5} + \cdots\right)\cdot\cosh\left(t_2\frac{x^2}{2} + t_4\frac{x^4}{4} + t_6\frac{x^6}{6} + \cdots\right).$$

124. (a) [2] Let $f_k(n)$ denote the number of permutations $w \in \mathfrak{S}_n$ with k inversions. Show combinatorially that for $n \geq k$,

$$f_k(n+1) = f_k(n) + f_{k-1}(n+1).$$

(b) [1+] Deduce from (a) that for $n \geq k$, $f_k(n)$ is a polynomial in n of degree k and leading coefficient $1/k!$. For instance, $f_2(n) = \frac{1}{2}(n+1)(n-2)$ for $n \geq 2$.

(c) [2+] Let $g_k(n)$ be the polynomial that agrees with $f_k(n)$ for $n \geq k$. Find $\Delta^j g_k(-n)$; that is, find the coefficients a_j in the expansion

$$g_k(-n) = \sum_{j=0}^{k} a_j \binom{n}{j}.$$

125. [2+]* Find the number $f(n)$ of binary sequences $w = a_1 a_2 \cdots a_k$ (where k is arbitrary) such that $a_1 = 1$, $a_k = 0$, and $\mathrm{inv}(w) = n$. For instance, $f(4) = 5$, corresponding to the sequences 10000, 11110, 10110, 10010, 1100. How many of these sequences have exactly j 1's?

126. [2+]* Show that

$$\sum_{w} q^{\mathrm{inv}(w)} = q^n \prod_{j=0}^{n-1} (1 + q^2 + q^4 + \cdots + q^{4j}),$$

where w ranges over all fixed-point free involutions in $\mathfrak{S}_{2n}$, and where $\mathrm{inv}(w)$ denotes the number of inversions of w. Give a simple combinatorial proof analogous to the proof of Corollary 1.3.13.

127. [2]

(a) Let $w \in \mathfrak{S}_n$, and let $R(w)$ be the set of positions of the records (or left-to-right maxima) of w. For instance, $R(3265174) = \{1, 3, 6\}$. For any finite set S of positive integers, set $x^S = \prod_{i \in S} x_i$. Show that

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} x^{R(w)} = x_1(x_2 + q)(x_3 + q + 1) \cdots (x_n + q + q^2 + \cdots + q^{n-1}). \quad (1.135)$$

(b) Let $V(w)$ be the set of the records themselves, e.g., $V(3265174) = \{3, 6, 7\}$. Show that

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} x^{V(w)} = (x_1 + q + q^2 + \cdots + q^{n-1})(x_2 + q + q^2 + \cdots + q^{n-2}) \cdots (x_{n-1} + q) x_n. \quad (1.136)$$

128. (a) [2] A permutation $a_1 \cdots a_n$ of $[n]$ is called *indecomposable* or *connected* if n is the least positive integer j for which $\{a_1, a_2, \ldots, a_j\} = \{1, 2, \ldots, j\}$. Let $f(n)$ be the number of indecomposable permutations of $[n]$, and set $F(x) = \sum_{n \geq 0} n! x^n$. Show that

$$\sum_{n \geq 1} f(n) x^n = 1 - \frac{1}{F(x)}. \quad (1.137)$$

(b) [2+] If $a_1 \cdots a_n$ is a permutation of $[n]$, then a_i is called a *strong fixed point* if (1) $j < i \Rightarrow a_j < a_i$, and (2) $j > i \Rightarrow a_j > a_i$ (so in particular $a_i = i$). Let $g(n)$ be the number of permutations of $[n]$ with no strong fixed points. Show that

$$\sum_{n \geq 0} g(n) x^n = \frac{F(x)}{1 + xF(x)}.$$

(c) [2+] A permutation $w \in \mathfrak{S}_n$ is *stabilized-interval-free* (SIF) if there does not exist $1 \leq i < j \leq n$ for which $w \cdot [i,j] = [i,j]$ (as sets). For instance, 615342 fails to be SIF since $w \cdot [3,5] = [3,5]$. Let $h(n)$ be the number of SIF permutations $w \in \mathfrak{S}_n$, and set

$$H(x) = \sum_{n\geq 0} h(n)x^n = 1 + x + x^2 + 2x^3 + 7x^4 + 34x^5 + 206x^6 + \cdots .$$

Show that

$$H(x) = \frac{x}{\left(\sum_{n\geq 0} n!x^{n+1}\right)^{\langle -1\rangle}},$$

where $^{\langle -1\rangle}$ denotes compositional inverse (§5.4). Equivalently, by the Lagrange inversion formula (Theorem 5.4.2), $H(x)$ is uniquely defined by the condition

$$[x^{n-1}]H(x)^n = n!, \quad n \geq 1.$$

(d) [2+] A permutation $w \in \mathfrak{S}_n$ is called *simple* if it maps no interval $[i,j]$ of size $1 < j - i + 1 < n$ into another such interval. For instance, 3157462 is not simple, since it maps $[3,6]$ into $[4,7]$ (as sets). Let $k(n)$ be the number of simple permutations $w \in \mathfrak{S}_n$, and set

$$K(x) = \sum_{n\geq 1} k(n)x^n = x + 2x^2 + 2x^4 + 6x^5 + 46x^7 + 338x^8 + \cdots .$$

Show that

$$K(x) = \frac{2}{1+x} - \left(\sum_{n\geq 1} n!x^n\right)^{\langle -1\rangle}.$$

129. (a) [2]* Let $f_k(n)$ be the number of indecomposable permutations $w \in \mathfrak{S}_n$ with k inversions. Generalizing equation (1.137), show that

$$\sum_{n\geq 1} f_k(n)q^k x^n = 1 - \frac{1}{F(q,x)},$$

where $F(q,x) = \sum_{n\geq 0} (\boldsymbol{n})!x^n$. As usual, $(\boldsymbol{n})! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$.

(b) [2] Write $1/F(q,x) = \sum_{n\geq 0} g_n(q)x^n$, where $g_n(q) \in \mathbb{Z}[q]$. Show that $\sum_{n\geq 0} g_n(q)$ is a well-defined formal power series, even though it makes no sense to substitute directly $x = 1$ in $1/F(q,x)$.

(c) [3] Write $1/F(q,x)$ in a form where it does make sense to substitute $x = 1$.

130. [2+] Let $u(n)$ be the number of permutations $w = a_1 \cdots a_n \in \mathfrak{S}_n$ such that $a_{i+1} \neq a_i \pm 1$ for $1 \leq i \leq n-1$. Equivalently, $f(n)$ is the number of ways to place n nonattacking kings on an $n \times n$ chessboard, no two on the same file or rank. Set

$$U(x) = \sum_{n\geq 0} u(n)x^n = 1 + x + 2x^4 + 14x^5 + 90x^6 + 646x^7 + 5242x^8 + \cdots .$$

Show that

$$U(x) = F\left(\frac{x(1-x)}{1+x}\right), \tag{1.138}$$

where $F(x) = \sum_{n\geq 0} n!x^n$ as in Exercise 1.128.

131. [2+]* An n-dimensional cube K_n has $2n$ facets (or $(n-1)$-dimensional faces), which come in n antipodal pairs. A *shelling* of K_n is equivalent to a linear ordering $F_1, F_2, \dots, F_{2n}$ of its facets such that for all $1 \leq i \leq n-1$, the set $\{F_1, \dots, F_{2i}\}$ does not consist of i antipodal pairs. Let $f(n)$ be the number of shellings of K_n. Show that

$$\sum_{n\geq 1} f(n)\frac{x^n}{n!} = 1 - \left(\sum_{n\geq 0}(2n)!\frac{x^n}{n!}\right)^{-1}.$$

132. [1+]* Let $w \in \mathfrak{S}_n$. Which of the following items doesn't belong?

- $\mathrm{inv}(w) = 0$
- $\mathrm{maj}(w) = 0$
- $\mathrm{des}(w) = 0$
- $\mathrm{maj}(w) = \mathrm{des}(w) = \mathrm{inv}(w)$
- $D(w) = \emptyset$
- $c(w) = n$ (where $c(w)$ denotes the number of cycles of w)
- $w^5 = w^{12} = 1$

133. (a) [2+] Let $A_n(x)$ be the Eulerian polynomial. Give a combinatorial proof that $\frac{1}{2}A_n(2)$ is equal to the number of *ordered* set partitions (i.e., partitions whose blocks are linearly ordered) of an n-element set.

(b) [2+]* More generally, show that

$$\frac{A_n(x)}{x} = \sum_{k=0}^{n-1}(n-k)!S(n, n-k)(x-1)^k.$$

Note that $(n-k)!S(n,n-k)$ is the number of ordered partitions of an n-set into $n-k$ blocks.

134. [3–] Show that

$$A_n(x) = \sum_w x^{1+\mathrm{des}(w)}(1+x)^{n-1-2\mathrm{des}(w)},$$

where w ranges over all permutations in $\mathfrak{S}_n$ with no proper double descents (as defined in Exercise 1.61) and with no descent at the end. For instance, when $n = 4$ the permutations are 1234, 1324, 1423, 2134, 2314, 2413, 3124, 3412, 4123.

135. (a) [2] Let $A_n(x)$ be the Eulerian polynomial. Show that

$$A_n(-1) = \begin{cases} (-1)^{(n+1)/2}E_n, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

(b) [3–] Give a combinatorial proof of (a) when n is odd.

136. [2+] What sequence $\boldsymbol{c} = (c_1, \dots, c_n) \in \mathbb{N}^n$ with $\sum ic_i = n$ maximizes the number of $w \in \mathfrak{S}_n$ of type $\boldsymbol{c}$? For instance, when $n = 4$ the maximizing sequence is $(1, 0, 1, 0)$.

137. [3–] Let ℓ be a prime number and write $n = a_0 + a_1\ell + a_2\ell^2 + \cdots$, with $0 \leq a_i < \ell$ for all $i \geq 0$. Let $\kappa_\ell(n)$ denote the number of sequences $\boldsymbol{c} = (c_1, c_2, \dots, c_n) \in \mathbb{N}^n$ with $\sum ic_i = n$, such that the number of permutations $w \in \mathfrak{S}_n$ of type $\boldsymbol{c}$ is relatively prime to ℓ. Show that

$$\kappa_\ell(n) = p(a_0) \prod_{i \geq 1} (a_i + 1),$$

where $p(a_0)$ is the number of partitions of a_0. In particular, the number of $\boldsymbol{c}$ such that an odd number of $w \in \mathfrak{S}_n$ have type $\boldsymbol{c}$ is 2^b, where $\lfloor n/2 \rfloor$ has b 1's in its binary expansion.

138. [2+]* Find a simple formula for the number of alternating permutations $a_1a_2 \cdots a_{2n} \in \mathfrak{S}_{2n}$ satisfying $a_2 < a_4 < a_6 < \cdots < a_{2n}$.

139. [2+] An *even tree* is a (rooted) tree such that every vertex has an even number of children. (Such a tree must have an odd number of vertices.) Note that these are *not* plane trees, i.e., we don't linearly order the subtrees of a vertex. Express the number $g(2n + 1)$ of increasing even trees with $2n + 1$ vertices in terms of Euler numbers. Use generating functions.

140. [3–] Define a *simsun permutation* to be a permutation $w \in \mathfrak{S}_n$ such that w has no proper double descents (as defined in Exercise 1.61(c)) and such that for all $0 \leq k \leq n - 1$, if we remove $n, n - 1, \cdots, n - k$ from w (written as a word) then the resulting permutation also has no proper double descents. For instance, $w = 3241$ is not simsun since if we remove 4 from w we obtain 321, which has a proper double descent. Show that the number of simsun permutations in $\mathfrak{S}_n$ is equal to the Euler number E_{n+1}.

141. (a) [2+] Let $E_{n,k}$ denote the number of alternating permutations of $[n + 1]$ with first term $k + 1$. For instance, $E_{n,n} = E_n$. Show that

$$E_{0,0} = 1, \quad E_{n,0} = 0 \ (n \geq 1), \quad E_{n+1,k+1} = E_{n+1,k} + E_{n,n-k} \ (n \geq k \geq 0). \quad (1.139)$$

Note that if we place the $E_{n,k}$'s in the triangular array

$$\begin{array}{ccccccccc}
 & & & & E_{00} & & & & \\
 & & & E_{10} & \rightarrow & E_{11} & & & \\
 & & E_{22} & \leftarrow & E_{21} & \leftarrow & E_{20} & & \\
 & E_{30} & \rightarrow & E_{31} & \rightarrow & E_{32} & \rightarrow & E_{33} & \\
E_{44} & \leftarrow & E_{43} & \leftarrow & E_{42} & \leftarrow & E_{41} & \leftarrow & E_{40} \\
 & & & & \cdots & & & &
\end{array} \quad (1.140)$$

and read the entries in the direction of the arrows from top-to-bottom (the so-called *boustrophedon* or *ox-plowing* order), then the first number read in each row is 0, and each subsequent entry is the sum of the previous entry and the entry above in the previous row. The first seven rows of the array are as follows:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & \rightarrow & 1 & & & & & \\
 & & & & 1 & \leftarrow & 1 & \leftarrow & 0 & & & & \\
 & & & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 2 & & & \\
 & & 5 & \leftarrow & 5 & \leftarrow & 4 & \leftarrow & 2 & \leftarrow & 0 & & \\
 & 0 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 14 & \rightarrow & 16 & \rightarrow & 16 & \\
61 & \leftarrow & 61 & \leftarrow & 56 & \leftarrow & 46 & \leftarrow & 32 & \leftarrow & 16 & \leftarrow & 0. \\
 & & & & & & \cdots & & & & & &
\end{array}$$

(b) [3–] Define

$$[m,n] = \begin{cases} m, & m+n \text{ odd} \\ n, & m+n \text{ even.} \end{cases}$$

Show that

$$\sum_{m\geq 0}\sum_{n\geq 0} E_{m+n,[m,n]} \frac{x^m}{m!}\frac{y^n}{n!} = \frac{\cos x + \sin x}{\cos(x+y)}. \tag{1.141}$$

142. [3–] Define polynomials $f_n(a)$ for $n \geq 0$ by $f_0(a) = 1$, $f_n(0) = 0$ if $n \geq 1$, and $f_n'(a) = f_{n-1}(1-a)$. Thus

$$\begin{aligned}
f_1(a) &= a \\
f_2(a) &= \frac{1}{2}(-a^2+2a) \\
f_3(a) &= \frac{1}{3!}(-a^3+3a) \\
f_4(a) &= \frac{1}{4!}(a^4-4a^3+8a) \\
f_5(a) &= \frac{1}{5!}(a^5-10a^3+25a) \\
f_6(a) &= \frac{1}{6!}(-a^6+6a^5-40a^3+96a).
\end{aligned}$$

Show that $\sum_{n\geq 0} f_n(1)x^n = \sec x + \tan x$.

143. (a) [2–] Let $\mathrm{fix}(w)$ denote the number of fixed points (cycles of length 1) of the permutation $w \in \mathfrak{S}_n$. Show that

$$\sum_{w\in\mathfrak{S}_n} \mathrm{fix}(w) = n!.$$

Try to give a combinatorial proof, a generating function proof, and an algebraic proof.

(b) [3+] Let Alt_n (respectively, Ralt_n) denote the set of alternating (respectively, reverse alternating) permutations $w \in \mathfrak{S}_n$. Define

$$\begin{aligned} f(n) &= \sum_{w \in \mathrm{Alt}_n} \mathrm{fix}(w) \\ g(n) &= \sum_{w \in \mathrm{Ralt}_n} \mathrm{fix}(w). \end{aligned}$$

Show that

$$\begin{aligned} f(n) &= \begin{cases} E_n - E_{n-2} + E_{n-4} - \cdots + (-1)^{(n-1)/2} E_1, & n \text{ odd} \\ E_n - 2E_{n-2} + 2E_{n-4} - \cdots + (-1)^{(n-2)/2} 2E_2 + (-1)^{n/2}, & n \text{ even.} \end{cases} \\ g(n) &= \begin{cases} E_n - E_{n-2} + E_{n-4} - \cdots + (-1)^{(n-1)/2} E_1, & n \text{ odd} \\ E_n - (-1)^{n/2}, & n \text{ even.} \end{cases} \end{aligned}$$

144. (a) [2] Let

$$F(x) = 2 \sum_{n \geq 0} q^n \frac{\prod_{j=1}^{n} (1 - q^{2j-1})}{\prod_{j=1}^{2n+1} (1 + q^j)},$$

where $q = \left(\frac{1-x}{1+x}\right)^{2/3}$. Show that $F(x)$ is well-defined as a formal power series. Note that $q(0) = 1 \neq 0$, so some special argument is needed.

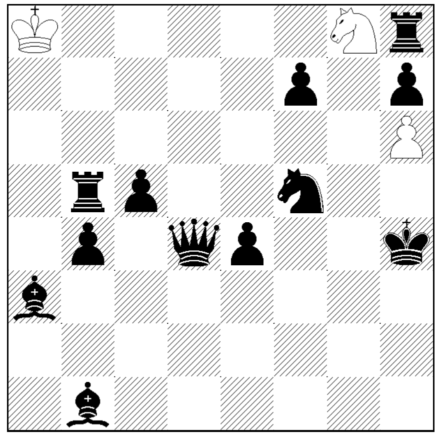
(b) [3+] Let $F(x)$ be defined by (a), and write

$$F(x) = \sum_{n \geq 0} f(n) x^n = 1 + x + x^2 + 2x^3 + 5x^4 + 17x^5 + 72x^6 + 367x^7 + 2179x^8 + \cdots.$$

Show that $f(n)$ is equal to the number of alternating fixed-point free involutions in $\mathfrak{S}_{2n}$, i.e., the number of permutations $w \in \mathfrak{S}_{2n}$ that are alternating permutations and have n cycles of length two. For instance, when $n = 3$ we have the two permutations 214365 and 645321, and when $n = 4$ we have the five permutations 21436587, 21867453, 64523187, 64827153, and 84627351.

145. [3–] Solve the following chess problem, where the condition "serieshelpmate" is defined in Exercise 1.11(c).

A. Karttunen, 2006



Serieshelpmate in 9: how many solutions?

146. [2+] Let $f_k(n)$ denote the number of permutations $w \in \mathfrak{S}_n$ such that

$$D(w) = \{k, 2k, 3k, \ldots\} \cap [n-1],$$

as in equation (1.58). Let $1 \le i \le k$. Show that

$$\sum_{m \ge 0} f_k(mk+i) \frac{x^{mk+i}}{(mk+i)!} = \frac{\sum_{m \ge 0} (-1)^m \frac{x^{mk+i}}{(mk+i)!}}{\sum_{m \ge 0} (-1)^m \frac{x^{mk}}{(mk)!}}.$$

Note that when $i = k$ we can add 1 to both sides and obtain equation (1.59).

147. [2+] Call two permutations $u, v \in \mathfrak{S}_n$ *equivalent* if their min-max trees $M(u)$ and $M(v)$ are isomorphic as *unlabelled* binary trees. This notion of equivalence is clearly an equivalence relation. Show that the number of equivalence classes is the Motzkin number M_{n-1} defined in Exercise 6.37 and further explicated in Exercise 6.38.

148. [2+] Let $\Phi_n = \Phi_n(c, d)$ denote the *cd*-index of $\mathfrak{S}_n$, as defined in Theorem 1.6.3. Thus $c = a + b$ and $d = ab + ba$. Let $S \subseteq [n-1]$, and let u_S be the variation of S as defined by equation (1.60). Show that

$$\Phi_n(a + 2b, ab + ba + 2b^2) = \sum_{S \subseteq [n-1]} \alpha(S) u_S,$$

where $\alpha(S)$ is given by equation (1.31).

149. [3–] If $F(x)$ is any power series with noncommutative coefficients such that $F(0) = 0$, then define $(1 - F(x))^{-1}$ to be the unique series $G(x)$ satisfying

$$(1 - F(x))G(x) = G(x)(1 - F(x)) = 1.$$

Equivalently, $G(x) = 1 + F(x) + F(x)^2 + \cdots$. Show that

$$\sum_{n\geq 1} \Phi_n(c,d)\frac{x^n}{n!} = \frac{\sinh(a-b)x}{a-b}\left[1 - \frac{1}{2}\left(\frac{c\cdot\sinh(a-b)x}{a-b} - \cosh(a-b)x + 1\right)\right]^{-1}. \tag{1.142}$$

Note that the series on the right involves only *even* powers of $a-b$. Since $(a-b)^2 = c^2 - 2d$, it follows that the coefficients of this series are indeed polynomials in c and d.

150. (a) [3–]* Let $f(n)$ (respectively, $g(n)$) be the total number of c's (respectively, d's) that appear when we write the cd-index $\Phi_n(c,d)$ as a sum of monomials. For instance, $\Phi_4(c,d) = c^3 + 2cd + 2dc$, so $f(4) = 7$ and $g(4) = 4$. Show using generating functions that $f(n) = 2E_{n+1} - (n+1)E_n$ and $g(n) = nE_n - E_{n+1}$.

    (b) [5–] Is there a combinatorial proof?

151. [3–] Let μ be a monomial of degree $n-1$ in the noncommuting variables c, d, where $\deg(c) = 1$ and $\deg(d) = 2$. Show that $[\mu]\Phi_n(c,d)$ is the number of sequences $\mu = \nu_0, \nu_1, \ldots, \nu_{n-1} = 1$, where ν_i is obtained from ν_{i-1} by removing a c or changing a d to c. For instance, if $\mu = dcc$ there are three sequences: $(dcc, ccc, cc, c, 1)$, $(dcc, dc, cc, c, 1)$, $(dcc, dc, d, c, 1)$.

152. [3–] Continue the notation from the previous exercise. Replace each c in μ with 0, each d with 10, and remove the final 0. We get the characteristic vector of a set $S_\mu \subseteq [n-2]$. For instance, if $\mu = cd^2c^2d$ then we get the characteristic vector 01010001 of the set $S_\mu = \{2,4,8\}$. Show that $[\mu]\Phi_n(c,d)$ is equal to the number of simsun permutations (defined in Exercise 1.140) in $\mathfrak{S}_{n-1}$ with descent set S_μ.

153. (a) [2] Let $f(n)$ denote the coefficient of d^n in the cd-index Φ_{2n+1}. Show that $f(n) = 2^{-n}E_{2n+1}$.

    (b) [3] Show that $f(n)$ is the number of permutations w of the multiset $\{1^2, 2^2, \ldots, (n+1)^2\}$ beginning with 1 such that between the two occurrences of i $(1 \leq i \leq n)$ there is exactly one occurrence of $i+1$. For instance, $f(2) = 4$, corresponding to 123123, 121323, 132312, 132132.

154. (a) [1+] Let $F(x) = \sum_{n\geq 0} f(n)x^n/n!$. Show that

$$e^{-x}F(x) = \sum_{n\geq 0}[\Delta^n f(0)]x^n/n!.$$

    (b) [2] Find the unique function $f : \mathbb{P} \to \mathbb{C}$ satisfying $f(1) = 1$ and $\Delta^n f(1) = f(n)$ for all $n \in \mathbb{P}$.

    (c) [2] Generalize (a) by showing that

$$e^{-x}F(x+t) = \sum_{n\geq 0}\sum_{k\geq 0}\Delta^n f(k)\frac{x^n}{n!}\frac{t^k}{k!}.$$

155. (a) [1+] Let $F(x) = \sum_{n\geq 0} f(n)x^n$. Show that

$$\frac{1}{1+x}F\left(\frac{x}{1+x}\right) = \sum_{n\geq 0}[\Delta^n f(0)]x^n.$$

(b) [2+] Find the unique functions $f, g : \mathbb{N} \to \mathbb{C}$ satisfying $\Delta^n f(0) = g(n)$, $\Delta^{2n} g(0) = f(n)$, $\Delta^{2n+1} g(0) = 0$, $f(0) = 1$.

(c) [2+] Find the unique functions $f, g : \mathbb{N} \to \mathbb{C}$ satisfying $\Delta^n f(1) = g(n)$, $\Delta^{2n} g(0) = f(n)$, $\Delta^{2n+1} g(0) = 0$, $f(0) = 1$.

156. [2+] Let A be the abelian group of all polynomials $p : \mathbb{Z} \to \mathbb{C}$ such that $D^k p : \mathbb{Z} \to \mathbb{Z}$ for all $k \in \mathbb{N}$. (D^k denotes the kth derivative, and $D^0 p = p$.) Then A has a basis of the form $p_n(x) = c_n\binom{x}{n}$, $n \in \mathbb{N}$, where c_n is a constant depending only on n. Find c_n explicitly.

157. [2] Let λ be a complex number (or indeterminate), and let

$$y = 1 + \sum_{n\geq 1} f(n)x^n, \quad y^\lambda = \sum_{n\geq 0} g(n)x^n.$$

Show that

$$g(n) = \frac{1}{n}\sum_{k=1}^{n}[k(\lambda+1) - n]f(k)g(n-k), \quad n \geq 1.$$

This formula affords a method of computing the coefficients of y^λ much more efficiently than using (1.5) directly.

158. [2+] Let $f_1, f_2, \ldots$ be a sequence of complex numbers. Show that there exist unique complex numbers $a_1, a_2, \ldots$ such that

$$F(x) := 1 + \sum_{n\geq 1} f_n x^n = \prod_{i\geq 1}(1 - x^i)^{-a_i}.$$

Set $\log F(x) = \sum_{n\geq 1} g_n x^n$. Find a formula for a_i in terms of the g_n's. What are the a_i's when $F(x) = 1 + x$ and $F(x) = e^{x/(1-x)}$?

159. [2] Let $F(x) = 1 + a_1x + \cdots \in K[[x]]$, where K is a field satisfying $\mathrm{char}(K) \neq 2$. Show that there exist unique series $A(x)$, $B(x)$ satisfying $A(0) = B(0) = 1$, $A(x) = A(-x)$, $B(x)B(-x) = 1$, and $F(x) = A(x)B(x)$. Find simple formulas for $A(x)$ and $B(x)$ in terms of $F(x)$.

160. (a) [2] Let $0 \leq j < k$. The (k, j)-*multisection* of the power series $F(x) = \sum_{n\geq 0} a_n x^n$ is defined by

$$\Psi_{k,j}F(x) = \sum_{m\geq 0} a_{km+j}x^{km+j}.$$

Let $\zeta = e^{2\pi i/k}$ (where $i^2 = -1$). Show that

$$\Psi_{k,j}F(x) = \frac{1}{k}\sum_{r=0}^{k-1}\zeta^{-jr}F(\zeta^r x).$$

(b) [2] As a simple application of (a), let $0 \le j < k$, and let $f(n,k,j)$ be the number of permutations $w \in \mathfrak{S}_n$ satisfying $\mathrm{maj}(w) \equiv j \pmod{k}$. Show that $f(n,k,j) = n!/k$ if $n \ge k$.

(c) [2+] Show that

$$f(k-1,k,0) = \frac{(k-1)!}{k} + \sum_{\zeta} \frac{1}{(1-\xi)^{k-1}},$$

where ξ ranges over all primitive kth roots of unity. Can this expression be simplified?

161. (a) [2]* Let $F(x) = a_0 + a_1 x + \cdots \in K[[x]]$, with $a_0 = 1$. For $k \ge 2$ define $F_k(x) = \Phi_{k,0}(x) = \sum_{m \ge 0} a_{km} x^{km}$. Show that for $n \ge 1$,

$$[x^{km}] \frac{F(x)}{\Phi_{k,0} F(x)} = 0.$$

(b) [2+] Let $\mathrm{char}\, K \neq 2$. Given $G(x) = 1 + H(x)$ where $H(-x) = -H(x)$ (i.e., $H(x)$ has only odd exponents), find the general solution $F(x) = 1 + a_1 x + \cdots$ to $F(x)/F_2(x) = G(x)$. Express your answer in the form $F(x) = \Phi(G(x))E(x)$, where $\Phi(x)$ is a function independent from $G(x)$, and where $E(x)$ ranges over some class $\mathcal{E}$ of power series, also independent from $G(x)$.

162. [3–] Let $g(x) \in \mathbb{C}[[x]]$, $g(0) = 0$, $g(x) = g(-x)$. Find all power series $f(x)$ such that $f(0) = 0$ and

$$\frac{f(x) + f(-x)}{1 - f(x)f(-x)} = g(x).$$

Express your answer as an explicit algebraic function of $g(x)$ and a power series $h(x)$ (independent from $g(x)$) taken from some class of power series.

163. Let $f(x) \in \mathbb{C}[[x]]$, $f(x) = x+$ higher order terms. We say that $F(x,y) \in \mathbb{C}[[x,y]]$ is a *formal group law* or *addition law* for $f(x)$ if $f(x+y) = F(f(x), f(y))$.

(a) [2–] Show that for every $f(x) \in \mathbb{C}[[x]]$ with $f(x) = x + \cdots$, there is a unique $F(x,y) \in \mathbb{C}[[x,y]]$ which is a formal group law for $f(x)$.

(b) [3] Show that $F(x,y)$ is a formal group law if and only if $F(x,y) = x+y+$ higher order terms, and

$$F(F(x,y),z) = F(x, F(y,z)).$$

(c) [2] Find $f(x)$ so that $F(x,y)$ is a formal group law for $f(x)$ in the following cases:

- $F(x,y) = x + y$
- $F(x,y) = x + y + xy$
- $F(x,y) = (x+y)/(1-xy)$
- $F(x,y) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

(d) [2+] Using equation (5.128), show that the formal group law for $f(x) = xe^{-x}$ is given by

$$F(x,y) = x + y - \sum_{n\geq 1}(n-1)^{n-1}\frac{x^n y + xy^n}{n!},$$

where we interpret $0^0 = 1$ in the summand indexed by $n = 1$.

(e) [3] Find the formal group law for the function

$$f(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}}.$$

164. [3–] Solve the following equation for the power series $F(x,y) \in \mathbb{C}[[x,y]]$:

$$(xy^2 + x - y)F(x,y) = xF(x,0) - y.$$

The point is to make sure that your solution has a power series expansion at $(0,0)$.

165. [2+] Find a simple description of the coefficients of the power series $F(x) = x + \cdots \in \mathbb{C}[[x]]$ satisfying the functional equation

$$F(x) = (1+x)F(x^2) + \frac{x}{1-x^2}.$$

166. [2] Let $n \in \mathbb{P}$. Find a power series $F(x) \in \mathbb{C}[[x]]$ satisfying $F(F(x))^n = 1 + F(x)^n$, $F(0) = 1$.

167. [2] Let $F(x) \in \mathbb{C}[[x]]$. Find a simple expression for the exponential generating function of the derivatives of $F(x)$, i.e.,

$$\sum_{n\geq 0} D^n F(x)\frac{t^n}{n!}, \tag{1.143}$$

where $D = d/dx$.

168. Let K be a field satisfying $\mathrm{char}(K) \neq 2$. If $A(x) = x + \sum_{n\geq 2} a_n x^n \in K[[x]]$, then let $A^{\langle -1\rangle}(x)$ denote the compositional inverse of A; that is, $A^{\langle -1\rangle}(A(x)) = A(A^{\langle -1\rangle}(x)) = x$.

(a) [3–] Show that we can specify $a_2, a_4, \ldots$ arbitrarily, and they then determine uniquely $a_3, a_5, \ldots$ so that $A(-A(-x)) = x$. For instance

$$\begin{aligned} a_3 &= a_2^2 \\ a_5 &= 3a_4a_2 - 2a_2^4 \\ a_7 &= 13a_2^6 - 18a_4a_2^3 + 2a_4^2 + 4a_2a_6. \end{aligned}$$

NOTE. Let $E(x) = A(-x)$. Then the conditions $A(x) = x + \cdots$ and $A(-A(-x)) = x$ are equivalent to $E(x) = -x + \cdots$ and $E(E(x)) = x$.

(b) [5–] What are the coefficients when a_{2n+1} is written as a polynomial in $a_2, a_4, \ldots$ as in (a)?

(c) [2+]* Show that $A(-A(-x)) = x$ if and only if there is a $B(x) = x + \sum_{n\geq 2} b_n x^n$ such that $A(x) = B^{\langle -1\rangle}(-B(-x))$.

(d) [2+] Show that if $A(-A(-x)) = x$, then there is a unique $B(x)$ as in (c) of the form $B(x) = x + \sum_{n\geq 1} b_{2n} x^{2n}$. For instance,

$$\begin{aligned} b_2 &= -\frac{1}{2}a_2 \\ b_4 &= \frac{1}{8}\left(5a_2^3 - 4a_4\right) \\ b_6 &= -\frac{1}{16}\left(49a_2^5 - 56a_2^2a_4 + 8a_6\right). \end{aligned}$$

(e) [5–] What are the coefficients when b_{2n} is written as a polynomial in $a_2, a_4, \dots$ as in (d)?

(f) [2+] For any $C(x) = x + c_2x^2 + c_3x^3 + \cdots$, show that there are unique power series

$$\begin{aligned} A(x) &= x + a_2x^2 + a_3x^3 + \cdots \\ D(x) &= x + d_3x^3 + d_5x^5 + \cdots \end{aligned}$$

such that $A(-A(-x)) = x$ and $C(x) = D(A(x))$. For instance,

$$\begin{aligned} a_2 &= c_2 \\ d_3 &= c_3 - c_2^2 \\ a_4 &= c_4 - 3c_3c_2 + 3c_2^3 \\ d_5 &= c_5 + 3c_2^2c_3 - 3c_2c_4 - c_2^4. \end{aligned}$$

(g) [2+] Find $A(x)$ and $D(x)$ as in (f) when $C(x) = -\log(1-x)$.

(h) [5–] What are the coefficients when a_{2n} and d_{2n+1} are written as a polynomial in $c_2, c_3, \dots$ as in (f)?

(i) [2+] Note that if $A(x) = x/(1+2x)$, then $A(-A(-x)) = x$. Show that

$$B^{\langle -1\rangle}(-B(-x)) = x/(1+2x)$$

if and only if $e^{-x}\sum_{n\geq 0} b_{n+1}x^n/n!$ is an even function of x (i.e., has only even exponents).

(j) [2+] Identify the coefficients b_{2n} of the unique $B(x) = x + \sum_{n\geq 1} b_{2n}x^{2n}$ satisfying $B^{\langle -1\rangle}(-B(-x)) = x/(1+2x)$.

169. [2] Find a closed-form expression for the following generating functions.

(a) $\displaystyle\sum_{n\geq 0}(n+2)^2x^n$

(b) $\displaystyle\sum_{n\geq 0}(n+2)^2\frac{x^n}{n!}$

(c) $\sum_{n\geq 0}(n+2)^2\binom{2n}{n}x^n$

170. (a) [2–] Given $a_0=\alpha$, $a_1=\beta$, $a_{n+1}=a_n+a_{n-1}$ for $n\geq 1$, compute $y=\sum_{n\geq 0}a_nx^n$.

(b) [2+] Given $a_0=1$ and $a_{n+1}=(n+1)a_n-\binom{n}{2}a_{n-2}$ for $n\geq 0$, compute $y=\sum_{n\geq 0}a_nx^n/n!$.

(c) [2] Given $a_0=1$ and $2a_{n+1}=\sum_{i=0}^{n}\binom{n}{i}a_ia_{n-i}$ for $n\geq 0$, compute $\sum_{n\geq 0}a_nx^n/n!$ and find a_n explicitly. Compare equation (1.55), where (in the notation of the present exercise), $a_1=1$ and the recurrence holds for $n\geq 1$.

(d) [3] Let $a_k(0)=\delta_{0k}$, and for $1\leq k\leq n+1$ let

$$a_k(n+1)=\sum_{j=0}^{n}\binom{n}{j}\sum_{\substack{2r+s=k-1\\ r,s\geq 0}}(a_{2r}(j)+a_{2r+1}(j))a_s(n-j).$$

Compute $A(x,t):=\sum_{k,n\geq 0}a_k(n)t^kx^n/n!$.

171. Given a sequence $a_0,a_1,\ldots$ of complex numbers, let $b_n=a_0+a_1+\cdots+a_n$.

(a) [1+]* Let $A(x)=\sum_{n\geq 0}a_nx^n$ and $B(x)=\sum_{n\geq 0}b_nx^n$. Show that

$$B(x)=\frac{A(x)}{1-x}.$$

(b) [2+] Let $A(x)=\sum_{n\geq 0}a_n\frac{x^n}{n!}$ and $B(x)=\sum_{n\geq 0}b_n\frac{x^n}{n!}$. Show that

$$B(x)=\left(I(e^{-x}A'(x))+a_0\right)e^x, \tag{1.144}$$

where I denotes the *formal integral*, i.e.,

$$I\left(\sum_{n\geq 0}c_nx^n\right)=\sum_{n\geq 0}c_n\frac{x^{n+1}}{n+1}=\sum_{n\geq 1}c_{n-1}\frac{x^n}{n}.$$

172. [3–] The *Legendre polynomial* $P_n(x)$ is defined by

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n\geq 0}P_n(x)t^n.$$

Show that $(1-x)^nP_n((1+x)/(1-x))=\sum_{k=0}^{n}\binom{n}{k}^2x^k$.

173. [2+] Find simple closed expressions for the coefficients of the power series (expanded about $x=0$):

(a) $\sqrt{\frac{1+x}{1-x}}$

(b) $2\left(\sin^{-1}\frac{x}{2}\right)^2$

(c) $\sin(t\,\sin^{-1}x)$

(d) $\cos(t\,\sin^{-1}x)$

(e) $\sin(x)\sinh(x)$

(f) $\sin(x)\sin(\omega x)\sin(\omega^2 x)$, where $\omega = e^{2\pi i/3}$

(g) $\cos(\log(1+x))$ (express the answer as the real part of a complex number)

174. [1–] Find the order (number of elements) of the finite field $\mathbb{F}_2$.

175. [2+]* For $i,j \geq 0$ and $n \geq 1$, let $f_n(i,j)$ denote the number of pairs (V,W) of subspaces of $\mathbb{F}_q^n$ such that $\dim V = i$, $\dim W = j$, and $V \cap W = \{0\}$. Find a formula for $f_n(i,j)$ which is a power of q times a q-multinomial coefficient.

176. [2+] A sequence of vectors $v_1, v_2, \ldots$ is chosen uniformly and independently from $\mathbb{F}_q^n$. Let $E(n)$ be the expected value of k for which $v_1,\ldots,v_k$ span $\mathbb{F}_q^n$ but $v_1,\ldots,v_{k-1}$ don't span $\mathbb{F}_q^n$. For instance

$$\begin{aligned} E(1) &= \frac{q}{q-1} \\ E(2) &= \frac{q(2q+1)}{(q-1)(q+1)} \\ E(3) &= \frac{q(3q^3+4q^2+3q+1)}{(q-1)(q+1)(q^2+q+1)}. \end{aligned}$$

Show that

$$E(n) = \sum_{i=1}^{n} \frac{q^i}{q^i-1}.$$

177. (a) [2+]* Let $f(n,q)$ denote the number of matrices $A \in \mathrm{Mat}(n,q)$ satisfying $A^2 = 0$. Show that

$$f(n,q) = \sum_{2i+j=n} \frac{\gamma_n(q)}{q^{i(i+2j)}\gamma_i(q)\gamma_j(q)},$$

where $\gamma_m(q) = \#\mathrm{GL}(m,q)$. (The sum ranges over all pairs $(i,j) \in \mathbb{N}\times\mathbb{N}$ satisfying $2i+j=n$.)

(b) [2]* Write $f(n,q) = g(n,q)(q-1)^k$ so that $g(n,1) \neq 0, \infty$. Thus $f(n,q)$ may be regarded as a q-analogue of $g(n,1)$. Show that

$$\sum_{n\geq 0} g(n,1)\frac{x^n}{n!} = e^{x^2+x}.$$

(c) [5–] Is there an intuitive explanation of why $f(n,q)$ is a "good" q-analogue of $g(n,1)$?

178. [2+]* Let $f(n)$ be the number of pairs (A, B) of matrices in $\mathrm{Mat}(n, q)$ satisfying $AB = 0$. Show that

$$f(n) = \sum_{k=0}^{n} q^{n(n-k)} \binom{\boldsymbol{n}}{\boldsymbol{k}} (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}).$$

179. [2–]* How many pairs (A, B) of matrices in $\mathrm{Mat}(n, q)$ satisfy $A + B = AB$?

180. [5–] How many matrices $A \in \mathrm{Mat}(n, q)$ have square roots, i.e., $A = B^2$ for some $B \in \mathrm{Mat}(n, q)$? The $q = 1$ situation is Exercise 5.11(a).

181. [2]* Find a simple formula for the number $f(n)$ of matrices $A = (A_{ij}) \in \mathrm{GL}(n, q)$ such that $A_{11} = A_{1n} = A_{n1} = A_{nn} = 0$.

182. [2+] Let $f(n, q)$ denote the number of matrices $A = (A_{ij}) \in \mathrm{GL}(n, q)$ such that $A_{ij} \neq 0$ for all i, j. Let $g(n, q)$ denote the number of matrices $B = (B_{ij}) \in \mathrm{GL}(n-1, q)$ such that $B_{ij} \neq 1$ for all i, j. Show that

$$f(n, q) = (q-1)^{2n-1} g(n, q).$$

183. [2] Prove the identity

$$\frac{1}{1 - qx} = \prod_{d \geq 1} \left(1 - x^d\right)^{-\beta(d)}, \tag{1.145}$$

where $\beta(d)$ is given by equation (1.103).

184. (a) [2]* Let $f_q(n)$ denote the number of monic polynomials $f(x)$ of degree n over $\mathbb{F}_q$ that do not have a zero in $\mathbb{F}_q$, i.e., for all $\alpha \in \mathbb{F}_q$ we have $f(\alpha) \neq 0$. Find a simple formula for $F(x) = \sum_{n \geq 0} f_q(n) x^n$. Your answer should not involve any infinite sums or products.

NOTE. The constant polynomials $f(x) = \beta$ for $0 \neq \beta \in \mathbb{F}_q$ are included in the enumeration, but not the polynomial $f(x) = 0$.

(b) [2]* Use (a) to find a simple explicit formula for $f(n, q)$ when n is sufficiently large (depending on q).

185. (a) [1]* Show that the number of monic polynomials of degree n over $\mathbb{F}_q$ is q^n.

(b) [2+] Recall that the *discriminant* of a polynomial $f(x) = (x - \theta_1) \cdots (x - \theta_n)$ is defined by

$$\mathrm{disc}(f) = \prod_{1 \leq i < j \leq n} (\theta_i - \theta_j)^2.$$

Show that the number $D(n, 0)$ of monic polynomials $f(x)$ over $\mathbb{F}_q$ with discriminant 0 (equivalently, $f(x)$ has an irreducible factor of multiplicity greater than 1) is q^{n-1}, $n \geq 2$.

(c) [2+] Generalize (a) and (b) as follows. Fix $k \geq 1$, and let X be any subset of $\mathbb{N}^k$ containing $(0, 0, \ldots, 0)$. If $f_1, \ldots, f_k$ is a sequence of monic polynomials over $\mathbb{F}_q$, then set $f = (f_1, \ldots, f_k)$ and $\deg(f) = (\deg(f_1), \ldots, \deg(f_k))$. Given an irreducible polynomial $p \in \mathbb{F}_q[x]$, let $\operatorname{mult}(p, f) = (\mu_1, \ldots, \mu_k)$, where μ_i is the multiplicity of p in f_i. Given $\beta \in \mathbb{N}^k$, let $N(\beta)$ be the number of k-tuples $f = (f_1, \ldots, f_k)$ of monic polynomials over $\mathbb{F}_q$ such that $\deg(f) = \beta$ and such that for any irreducible polynomial p over $\mathbb{F}_q$ we have $\operatorname{mult}(p, f) \in X$. By a straightforward generalization of Exercise 1.158 to the multivariate case, there are unique $a_\alpha \in \mathbb{Z}$ such that

$$F_X(x) := \sum_{\alpha \in X} x^\alpha = \prod_{\substack{\alpha \in \mathbb{N}^k \\ \alpha \neq (0,0,\ldots,0)}} (1 - x^\alpha)^{a_\alpha}, \qquad (1.146)$$

where if $\alpha = (\alpha_1, \ldots, \alpha_k)$ then $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$. Show that

$$\sum_{\beta \in \mathbb{N}^k} N(\beta) x^\beta = \prod_{\substack{\alpha \in \mathbb{N}^k \\ \alpha \neq (0,0,\ldots,0)}} (1 - q x^\alpha)^{a_\alpha}.$$

Note that if $k = 1$ and $X = \mathbb{N}$, then $N(\beta)$ is the total number of monic polynomials of degree β. We have $f_{\mathbb{N}}(x) = 1/(1-x)$ and $\sum_{\beta \in \mathbb{N}} N(\beta) x^\beta = 1/(1-qx) = \sum_{n \geq 0} q^n x^n$, agreeing with (a).

186. Deduce from Exercise 1.185(c) the following results.

(a) [2] The number $N_r(n)$ of monic polynomials $f \in \mathbb{F}_q[x]$ of degree n with no factor of multiplicity at least r is given by

$$N_r(n) = q^n - q^{n-r+1}, \quad n \geq r. \qquad (1.147)$$

Note that the case $r = 2$ is equivalent to (b)

(b) [2] Let $N(m, n)$ be the number of pairs (f, g) of monic relatively prime polynomials over $\mathbb{F}_q$ of degrees m and n. In other words, f and g have nonzero resultant. Then

$$N(m, n) = q^{m+n-1}, \quad m, n \geq 1. \qquad (1.148)$$

(c) [2+] A polynomial f over a field K is *powerful* if every irreducible factor of f occurs with multiplicity at least two. Let $P(n)$ be the number of powerful monic polynomials of degree n over $\mathbb{F}_q$. Show that

$$P(n) = q^{\lfloor n/2 \rfloor} + q^{\lfloor n/2 \rfloor - 1} - q^{\lfloor (n-1)/3 \rfloor}, \quad n \geq 2. \qquad (1.149)$$

187. (a) [3–] Let q be an odd prime power. Show that as f ranges over all monic polynomials of degree $n > 1$ over $\mathbb{F}_q$, $\operatorname{disc}(f)$ is just as often a nonzero square in $\mathbb{F}_q$ as a nonsquare.

(b) [2+] For $n > 1$ and $a \in \mathbb{F}_q$, let $D(n, a)$ denote the number of monic polynomials of degree n over $\mathbb{F}_q$ with discriminant a. Thus by Exercise 1.185(b) we have $D(n, 0) = q^{n-1}$. Show that if $(n(n-1), q-1) = 1$ (so $q = 2^m$) or $(n(n-1), q-1) = 2$ (so q is odd) then $D(n, a) = q^{n-1}$ for all $a \in \mathbb{F}_q$. (Here (r, s) denotes the greatest common divisor of r and s.)

(c) [5–] Investigate further the function $D(n, a)$ for general n and a.

188. [3] Give a direct proof of Corollary 1.10.11, i.e., the number of nilpotent matrices in $\mathrm{Mat}(n, q)$ is $q^{n(n-1)}$.

189. [3–] Let V be an $(m + n)$-dimensional vector space over $\mathbb{F}_q$, and let $V = V_1 \oplus V_2$, where $\dim V_1 = m$ and $\dim V_2 = n$. Let $f(m, n)$ be the number of nilpotent linear transformations $A \colon V \to V$ satisfying $A(V_1) \subseteq V_2$ and $A(V_2) \subseteq V_1$. Show that

$$f(m, n) = q^{m(n-1)+n(m-1)}(q^m + q^n - 1),$$

190. (a) [2] Let $\omega^*(n, q)$ denote the number of conjugacy classes in the group $\mathrm{GL}(n, q)$. Show that $\omega^*(n, q)$ is a polynomial in q satisfying $\omega^*(n, 1) = 0$. For instance,

$$\begin{aligned}
\omega^*(1, q) &= q - 1 \\
\omega^*(2, q) &= q^2 - 1 \\
\omega^*(3, q) &= q^3 - q \\
\omega^*(4, q) &= q^4 - q \\
\omega^*(5, q) &= q^5 - q^2 - q + 1 \\
\omega^*(6, q) &= q^6 - q^2 \\
\omega^*(7, q) &= q^7 - q^3 - q^2 + 1 \\
\omega^*(8, q) &= q^8 - q^3 - q^2 + q.
\end{aligned}$$

(b) [2+] Show that

$$\omega^*(n, q) = q^n - q^{\lfloor (n-1)/2 \rfloor} + O(q^{\lfloor (n-1)/2 \rfloor - 1}).$$

(c) [3–] Evaluate the polynomial values $\omega^*(n, 0)$ and $\omega^*(n, -1)$. When is $\omega^*(n, q)$ divisible by q^2?

191. [3–] Give a more conceptual proof of Proposition 1.10.2, i.e., the number $\omega(n, q)$ of orbits of $\mathrm{GL}(n, q)$ acting adjointly on $\mathrm{Mat}(n, q)$ is given by

$$\omega(n, q) = \sum_j p_j(n) q^j.$$

192. (a) [2]* Find a simple formula for the number of surjective linear transformations $A : \mathbb{F}_q^n \to \mathbb{F}_q^k$.

(b) [2]* Show that the number of $m \times n$ matrices of rank k over $\mathbb{F}_q$ is given by

$$\binom{\boldsymbol{m}}{\boldsymbol{k}}(q^n-1)(q^n-q)\cdots(q^n-q^{k-1}).$$

193. [2] Let p_n denote the number of projections $P \in \mathrm{Mat}(n,q)$, i.e., $P^2 = P$. Show that

$$\sum_{n\geq 0} p_n \frac{x^n}{\gamma_n} = \left(\sum_{k\geq 0} \frac{x^k}{\gamma(k)}\right)^2,$$

where as usual $\gamma(k) = \gamma(k,q) = \#\mathrm{GL}(k,q)$.

194. [2+] Let r_n denote the number of regular (or cyclic) $M \in \mathrm{Mat}(n,q)$, i.e., the characteristic and minimal polynomials of A are the same. Equivalently, there is a column vector $v \in \mathbb{F}_q^n$ such that the set $\{A^i v : i \geq 0\}$ spans $\mathbb{F}_q^n$ (where we set $A^0 = I$). Show that

$$\begin{aligned}\sum_{n\geq 0} r_n \frac{x^n}{\gamma(n)} &= \prod_{d\geq 1}\left(1+\frac{x^d}{(q^d-1)(1-(x/q)^d)}\right)^{\beta(d)}\\ &= \frac{1}{1-x}\prod_{d\geq 1}\left(1+\frac{x^d}{q^d(q^d-1)}\right)^{\beta(d)}.\end{aligned}$$

195. [2] A matrix A is *semisimple* if it can be diagonalized over the algebraic closure of the base field. Let s_n denote the number of semisimple matrices $A \in \mathrm{Mat}(n,q)$. Show that

$$\sum_{n\geq 0} s_n \frac{x^n}{\gamma(n,q)} = \prod_{d\geq 1}\left(\sum_{j\geq 0}\frac{x^{jd}}{\gamma(j,q^d)}\right)^{\beta(d)}.$$

196. (a) [2+] Generalize Proposition 1.10.15 as follows. Let $0 \leq k \leq n$, and let $f_k(n)$ be the number of matrices $A = (a_{ij}) \in \mathrm{GL}(n,q)$ satisfying $a_{11} + a_{22} + \cdots + a_{kk} = 0$. Then

$$f_k(n) = \frac{1}{q}\left(\gamma(n,q) + (-1)^k(q-1)q^{\frac{1}{2}k(2n-k-1)}\gamma(n-k,q)\right). \tag{1.150}$$

(b) [2+] Let H be any linear hyperplane in the vector space $\mathrm{Mat}(n,q)$. Find (in terms of certain data about H) a formula for $\#(\mathrm{GL}(n,q) \cap H)$.

197. [3] Let $f(n)$ be the number of matrices $A \in \mathrm{GL}(n,q)$ with zero diagonal (i.e., all diagonal entries are equal to 0). Show that

$$f(n) = q^{\binom{n-1}{2}-1}(q-1)^n \sum_{i=0}^{n}(-1)^i\binom{n}{i}\boldsymbol{(n-i)!}.$$

For instance,

$$\begin{aligned} f(1) &= 0 \\ f(2) &= (q-1)^2 \\ f(3) &= q(q-1)(q^4-4q^2+4q-1) \\ f(4) &= q^3(q-1)(q^8-q^6-5q^5+3q^4+11q^3-14q^2+6q-1). \end{aligned}$$

198. (a) [2+] Let $h(n,r)$ denote the number of $n \times n$ symmetric matrices of rank r over $\mathbb{F}_q$. Show that

$$h(n+1,r) = q^r h(n,r) + (q-1)q^{r-1}h(n,r-1) + (q^{n+1}-q^{r-1})h(n,r-2), \quad (1.151)$$

with the initial conditions $h(n,0) = 1$ and $h(n,r) = 0$ for $r > n$.

(b) [2] Deduce that

$$h(n,r) = \begin{cases} \displaystyle\prod_{i=1}^{s} \frac{q^{2i}}{q^{2i}-1} \cdot \prod_{i=0}^{2s-1}(q^{n-i}-1), & 0 \le r = 2s \le n \\ \displaystyle\prod_{i=1}^{s} \frac{q^{2i}}{q^{2i}-1} \cdot \prod_{i=0}^{2s}(q^{n-i}-1), & 0 \le r = 2s+1 \le n. \end{cases}$$

In particular, the number $h(n,n)$ of $n \times n$ invertible symmetric matrices over $\mathbb{F}_q$ is given by

$$h(n,n) = \begin{cases} q^{m(m-1)}(q-1)(q^3-1)\cdots(q^{2m-1}), & n = 2m-1 \\ q^{m(m+1)}(q-1)(q^3-1)\cdots(q^{2m-1}), & n = 2m. \end{cases}$$

199. (a) [3] Show that the following three numbers are equal:

- The number of symmetric matrices in $\mathrm{GL}(2n,q)$ with zero diagonal.
- The number of symmetric matrices in $\mathrm{GL}(2n-1,q)$.
- The number of skew-symmetric matrices ($A = -A^t$) in $\mathrm{GL}(2n,q)$.

(b) [5] Give a combinatorial proof of (a). (No combinatorial proof is known that two of these items are equal.)

200. [3] Let $C_n(q)$ denote the number of $n \times n$ upper-triangular matrices X over $\mathbb{F}_q$ satisfying $X^2 = 0$. Show that

$$\begin{aligned} C_{2n}(q) &= \sum_j \left[\binom{2n}{n-3j} - \binom{2n}{n-3j-1}\right] \cdot q^{n^2-3j^2-j} \\ C_{2n+1}(q) &= \sum_j \left[\binom{2n+1}{n-3j} - \binom{2n+1}{n-3j-1}\right] \cdot q^{n^2+n-3j^2-2j}. \end{aligned}$$

201. This exercise and the next show that simply-stated counting problems over $\mathbb{F}_q$ can have complicated solutions beyond the realm of combinatorics. (See also Exercise 4.39(a).)

(a) [3] Let

$$f(q) = \#\{(x,y,z) \in \mathbb{F}_q^3 \,:\, x+y+z=0,\ xyz=1\}.$$

Show that $f(q) = q + a - 2$, where:

- if $q \equiv 2 \pmod 3$ then $a = 0$,
- if $q \equiv 1 \pmod 3$ then a is the unique integer such that $a \equiv 1 \pmod 3$ and $a^2 + 27b^2 = 4q$ for some integer b.

(b) [2+] Let

$$g(q) = \#\{A \in \mathrm{GL}(3,q) \,:\, \mathrm{tr}(A) = 0,\ \det(A) = 1.\}$$

Express $g(q)$ in terms of the function $f(q)$ of part (a).

202. [4–] Let p be a prime, and let N_p denote the number of solutions modulo p to the equation $y^2 + y = x^3 - x$. Let $a_p = p - N_p$. For instance, $a_2 = -2$, $a_3 = 1$, $a_5 = 1$, $a_7 = -2$, etc. Show that if $p \neq 11$, then

$$\begin{aligned} a_p &= [x^p]x\prod_{n\geq 1}(1-x^n)^2(1-x^{11n})^2 \\ &= [x^p](x - 2x^2 - x^3 + 2x^4 + x^5 + 2x^6 - 2x^7 - 2x^9 + \cdots) \end{aligned}$$

203. [3] The following quotation is from Plutarch's *Table-Talk* VIII. 9, 732: "Chrysippus says that the number of compound propositions that can be made from only ten simple propositions exceeds a million. (Hipparchus, to be sure, refuted this by showing that on the affirmative side there are 103,049 compound statements, and on the negative side 310,952.)"

According to T. L. Heath, *A History of Greek Mathematics*, vol. 2, p. 245, "it seems impossible to make anything of these figures."

Can in fact any sense be made of Plutarch's statement?