SOLUTIONS TO EXERCISES

1. *Answer:* 2. There is strong evidence that human babies, chimpanzees, and even rats have an understanding of this problem. See S. Dehaene, *The Number Sense: How the Mind Creates Mathematics*, Oxford, New York, 1997 (pages 23–27, 52–56).

2. Here is one possible way to arrive at the answers. There may be other equally simple (or even simpler) ways to solve these problems.

   (a) $2^{10} - 2^5 = 992$

   (b) $\frac{1}{2}(7-1)! = 360$

   (c) $5 \cdot 5!$ (or $6! - 5!$) $= 600$

   (d) $\binom{6}{1} 4! + \binom{6}{2} 3! + \frac{1}{2}\binom{6}{3} 2!^2 = 274$

   (e) $\binom{6}{4} + \binom{6}{1}\binom{5}{2} + \frac{1}{3!}\binom{6}{2}\binom{4}{2} = 90$

   (f) $(6)_4 = 360$

   (g) $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 945$

   (h) $\binom{7}{2} + \binom{8}{3} + \binom{9}{1} = 86$

   (i) $\binom{11}{1,2,4,4} - \binom{8}{1,1,2,4} = 33810.$

   (j) $\binom{8+1}{4} = 126$

   (k) $2\binom{8}{1,3,4} + 3\binom{8}{2,3,3} + \binom{8}{2,2,4} = 2660$

   (l) $5! + \binom{5}{2}(5)_4 + \frac{1}{2}\binom{5}{1}\binom{4}{2}(5)_3 = 2220$

3. (a) Given any n-subset S of $[x+n+1]$, there is a largest k for which $\#(S \cap [x+k]) = k$. Given k, we can choose S to consist of any k-element subset in $\binom{x+k}{k}$ ways, together with $\{x+k+2, x+k+3, \ldots, x+n+1\}$.

   (b) *First proof.* Choose a subset of $[n]$ and circle one of its elements in $\sum k\binom{n}{k}$ ways. Alternatively, circle an element of $[n]$ in n ways, and choose a subset of what remains in 2^{n-1} ways.

   Second proof (not quite so combinatorial, but nonetheless instructive). Divide the identity by 2^n. It then asserts that the average size of a subset of $[n]$ is $n/2$. This follows since each subset can be paired with its complement.

(c) To give a non-combinatorial proof, simply square both sides of the identity (Exercise 1.8(a))

$$\sum_{n\geq 0}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}$$

and equate coefficients. The problem of giving a combinatorial proof was raised by P. Veress and solved by G. Hajos in the 1930s. For some published proofs, see D. J. Kleitman, *Studies in Applied Math.* **54** (1975), 289–292; M. Sved, *Math. Intelligencer*, **6**(4) (1984), 44–45; and V. De Angelis, *Amer. Math. Monthly* **113** (2006), 642–644.

(d) G. E. Andrews, *Discrete Math.* **11** (1975), 97–106.

(e) Given an n-element subset S of $[2n-1]$, associate with it the two n-element subsets S and $[2n]-S$ of $[2n]$.

(f) What does it mean to give a combinatorial proof of an identity with minus signs? The simplest (but not the only) possibility is to rearrange the terms so that all signs are positive. Thus we want to prove that

$$\sum_{k\text{ even}}\binom{n}{k} = \sum_{k\text{ odd}}\binom{n}{k},\quad n\geq 1. \tag{1.152}$$

Let $\mathcal{E}_n$ (respectively $\mathcal{O}_n$) denote the sets of all subsets of $[n]$ of even (respectively, odd) cardinality. The left-hand side of equation (1.152) is equal to $\#\mathcal{E}_n$, while the right-hand side is $\#\mathcal{O}_n$. Hence we want to give a bijection $\varphi\colon \mathcal{E}_n\to\mathcal{O}_n$. The definition of φ is very simple:

$$\varphi(S) = \begin{cases} S\cup\{n\}, & n\notin S\\ S-\{n\}, & n\in S.\end{cases}$$

Another way to look at this proof is to consider φ as an involution on all of $2^{[n]}$. Every orbit of φ has two elements, and their contributions to the sum $\sum_{S\subseteq[n]}(-1)^{\#S}$ cancel out, i.e., $(-1)^{\#S}+(-1)^{\#\varphi(S)}=0$. Hence φ is a *sign-reversing involution* as in the proof of Proposition 1.8.7.

(g) The left-hand side counts the number of triples (S,T,f), where $S\subseteq[n]$, $T\subseteq[n+1,2n]$, $\#S=\#T$, and $f\colon S\to[x]$. The right-hand side counts the number of triples (A,B,g), where $A\subseteq[n]$, $B\in\binom{[2n]-A}{n}$, and $g : A\to[x-1]$. Given (S,T,f), define (A,B,g) as follows: $A=f^{-1}([x-1])$, $B=([n]-S)\cup T$, and $g(i)=f(i)$ for $i\in[x-1]$.

(h) We have that $\binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{i}$ is the number of triples (α,β,γ), where (i) α is a sequence of $i+j+2$ letters a and b beginning with a and ending with b, with $i+1$ a's (and hence $j+1$ b's), (ii) $\beta=(\beta_1,\dots,\beta_{j+1})$ is a sequence of $j+1$ positive integers with sum $j+k+1$, and (iii) $\gamma=(\gamma_1,\dots,\gamma_{i+1})$ is a sequence of $i+1$ positive integers with sum $k+i+1$. Replace the rth a in α by the word $c^{\gamma_r}d$, and replace the rth b in α by the word $d^{\beta_r}c$. In this way we obtain a word δ in c,d of length $2n+4$ with $n+2$ c's and $n+2$ d's. This word begins with c and ends with

$d(dc)^m$ for some $m \geq 1$. Remove the prefix c and suffix $d(dc)^m$ from δ to obtain a word ϵ of length $2(n-m+1)$ with $n-m+1$ c's and $n-m+1$ d's. The map $(\alpha, \beta, \gamma) \mapsto \epsilon$ is easily seen to yield a bijective proof of (h). This argument is due to Roman Travkin (private communication, October 2007).

Example. Let $n=8$, $i=2$, $j=k=3$, $\alpha = abbaabb$, $\beta = (2,3,1,1)$, $\gamma = (2,3,1)$. Then

$$\delta = (c^2d)(d^2c)(d^3c)(c^3d)(cd)(dc)(dc),$$

so $\epsilon = cd^3cd^3c^4dc$.

NOTE. Almost any binomial coefficient identity can be proved nowadays automatically by computer. For an introduction to this subject, see M. Petkovšek, H. S. Wilf, and D. Zeilberger, *A=B*, A K Peters, Wellesley, MA, 1996. Of course it is still of interest to find elegant bijective proofs of such identities.

8. (a) We have $1/\sqrt{1-4x} = \sum_{n\geq 0} \binom{-1/2}{n}(-4)^n x^n$. Now

$$\begin{aligned} \binom{-1/2}{n}(-4)^n &= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)(-4)^n}{n!} \\ &= \frac{2^n \cdot 1 \cdot 3 \cdots (2n-1)}{n!} = \frac{(2n)!}{n!^2}. \end{aligned}$$

(b) Note that $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$, $n > 0$ (see Exercise 1.3(e)).

9. (b) While powerful methods exist for solving this type of problem (see Example 6.3.8), we give here a "naive" solution. Suppose the path has k steps of the form $(0,1)$, and therefore k $(1,0)$'s and $n-k$ $(1,1)$'s. These $n+k$ steps may be chosen in any order, so

$$f(n,n) = \sum_k \binom{n+k}{n-k,k,k} = \sum_k \binom{n+k}{2k}\binom{2k}{k}.$$

$$\begin{aligned} \Rightarrow \sum_{n\geq 0} f(n,n)x^n &= \sum_k \binom{2k}{k} \sum_{n\geq 0} \binom{n+k}{2k} x^n \\ &= \sum_k \binom{2k}{k} \frac{x^k}{(1-x)^{2k+1}} \\ &= \frac{1}{1-x}\left(1 - \frac{4x}{(1-x)^2}\right)^{-1/2}, \text{ by Exercise 1.8(a)} \\ &= \frac{1}{\sqrt{1-6x+x^2}}. \end{aligned}$$

10. Let the elements of S be $a_1 < a_2 < \cdots < a_{r+s}$. Then the multiset $\{a_1, a_2 - 2, a_3 - 4, \ldots, a_{r+s} - 2(r+s-1)\}$ consists of r odd numbers and s even numbers in $[2(n -$

$r-s+1)]$. Conversely we can recover S from any r odd numbers and s even numbers (allowing repetition) in $[2(n-r-s+1)]$. Hence

$$f(n,r,s) = \left(\!\!\binom{n-r-s+1}{r}\!\!\right)\left(\!\!\binom{n-r-s+1}{s}\!\!\right) = \binom{n-r}{s}\binom{n-s}{r}.$$

This result is due to Jim Propp, private communication dated 29 July 2006. Propp has generalized the result to any modulus $m \geq 2$ and has also given a q-analogue.

11. (a) Choose $m+n+1$ points uniformly and independently from the interval $[0,1]$. The integral is then the probability that the last chosen point u is greater than the first m of the other points and less than the next n points. There are $(m+n+1)!$ orderings of the points, of which exactly $m!n!$ of them have the first m chosen points preceding u and the next n following u. Hence

$$B(m+1,n+1) = \frac{m!\,n!}{(m+n+1)!}.$$

The function $B(x,y)$ for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$ is the *beta function.*

There are many more interesting examples of the combinatorial evaluation of integrals. Two of the more sophisticated ones are P. Valtr, *Discrete Comput. Geom.* **13** (1995), 637–643; and *Combinatorica* **16** (1996), 567–573.

(b) Choose $(1+r+s)n+2t\binom{n}{2}$ points uniformly and independently from $[0,1]$. Label the first n chosen points x, the next r chosen points y_1, etc., so that the points are labelled by the elements of M. Let P be the probability that the order of the points in $[0,1]$ is a permutation of M that we are counting. Then

$$\begin{aligned} P &= \frac{n!\,r!^n s!^n (2t)!^{\binom{n}{2}}}{((r+s+1)n+tn(n-1))!} f(n,r,s,t) \\ &= \int_0^1 \cdots \int_0^1 (x_1\cdots x_n)^r((1-x_1)\cdots(1-x_n))^s \prod_{1\leq i<j\leq n}(x_i-x_j)^{2t} dx_1\cdots dx_n. \end{aligned}$$

This integral is the famous *Selberg integral*; see e.g. G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge/New York, 1999 (Chapter 8), and P. J. Forrester and S. O. Warnaar, *Bull. Amer. Math. Soc.* **45** (2008), 489–534. The evaluation of this integral immediately gives equation (1.119). No combinatorial proof of (1.119) is known. Such a proof would be quite interesting since it would give a combinatorial evaluation of Selberg's integral.

(c) One solution is 1.Pa5 2.Pa4 3.Pa3 4.Ra4 5.Ra8 6.Paxb2 7.Pb1=B 8.Pe2 9.Pe3 10.Bxf5 11.Bxe6 12.Bc8 13.Pg3 14.Pg2, after which White plays Bh2 mate. We attach indeterminates to each of the Black moves as follows: 1.a_{12} 2.a_{12} 3.x 4.a_{24} 5.a_{24} 6.a_{23} 7.a_{23} 8.a_{13} 9.a_{13} 10.x 11.a_{34} 12.a_{34} 13.a_{14} 14.a_{14}. We also place an indeterminate x before Black's first move and after Black's last move. All solutions are then obtained by permutations of Black's 14 moves, together with x at the

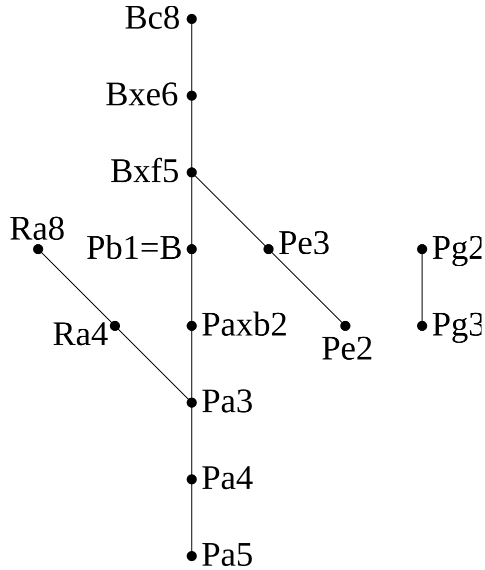


Figure 1.29: The solution poset for Exercise 1.11(c)

beginning and end, with the property that moves labelled by the same indeterminate must be played in the same order, and moves labelled a_{ij} must occur between the ith x and jth x. In the terminology of Chapter 3, the solutions correspond to the linear extensions of the poset shown in Figure 1.29. Hence the number of solutions is

$$f(4,0,0,1) = 54054.$$

For similar serieshelpmates (called *queue problems*) whose number of solutions has some mathematical significance, see Exercises 1.145, 6.23 and 7.18. Some references are given in the solution to Exercise 6.23. The present problem comes from the article R. Stanley, *Suomen Tehtäväniekat* **59**, no. 4 (2005), 193–203.

13. Let S consist of all p-tuples $(n_1, n_2, \ldots, n_p)$ of integers $n_i \in [a]$ such that not all the n_i's are equal. Hence $\#S = a^p - a$. Define two sequences in S to be equivalent if one is a cyclic shift of the other (clearly an equivalence relation). Since p is prime each equivalence class contains exactly p elements, and the proof follows. For additional results of this nature, see I. M. Gessel, in *Enumeration and Design (Waterloo, Ont., 1982)*, Academic Press, Toronto, ON, 1984, pp. 157–197, and G.-C. Rota and B. E. Sagan, *European J. Combin.* **1** (1980), 67–76.

14. (a) We use the well-known and easily proved fact that $(x+1)^p \equiv x^p + 1 \pmod{p}$, meaning that each coefficient of the polynomial $(x+1)^p - (x^p+1)$ is divisible by

p. Thus

$$\begin{aligned}(x+1)^n &= (x+1)^{\sum a_i p^i}\\ &\equiv \prod_i \left(x^{p^i}+1\right)^{a_i} \pmod p\\ &\equiv \prod_i \sum_{j=0}^{a_i} \binom{a_i}{j} x^{jp^i} \pmod p.\end{aligned}$$

The coefficient of x^m on the left is $\binom{n}{m}$ and on the right is $\binom{a_0}{b_0}\binom{a_1}{b_1}\cdots$. This congruence is due to F. E. A. Lucas, *Bull. Soc. Math. France* **6** (1878), 49–54.

(b) The binomial coefficient $\binom{n}{m}$ is odd if and only if the binary expansion of m is "contained" in that of n; that is, if m has a 1 in its ith binary digit, then so does n. Hence $\binom{n}{m}$ is odd for all $0 \le m \le n$ if and only if $n = 2^k - 1$. More generally, the number of odd coefficients of $(1+x)^n$ is equal to $2^{b(n)}$, where $b(n)$ is the number of 1's in the binary expansion of n. See Exercise 1.15 for some variations.

(c) Consider an $a \times p$ rectangular grid of squares. Choose pb of these squares in $\binom{pa}{pb}$ ways. We can choose the pb squares to consist of b entire rows in $\binom{a}{b}$ ways. Otherwise in at least two rows we will have picked between 1 and $p-1$ squares. For any choice of pb squares, cyclically shift the squares in each row independently. This partitions our choices into equivalence classes. Exactly $\binom{a}{b}$ of these classes contain one element; the rest contain a number of elements divisible by p^2.

(d) Continue the reasoning of (c). If a choice of pb squares contains fewer than $b-2$ entire rows, then its equivalence class has cardinality divisible by p^3. From this we reduce the problem to the case $a = 2$, $b = 1$. Now

$$\begin{aligned}\binom{2p}{p} &= \sum_{k=0}^{p} \binom{p}{k}^2\\ &= 2 + p^2 \sum_{k=1}^{p-1} \frac{(p-1)^2(p-2)^2\cdots(p-k+1)^2}{k!^2}\\ &\equiv 2 + p^2 \sum_{k=1}^{p-1} k^{-2} \pmod{p^3}.\end{aligned}$$

But as k ranges from 1 to $p-1$, so does k^{-1} modulo p. Hence

$$\sum_{k=1}^{p-1} k^{-2} \equiv \sum_{k=1}^{p-1} k^2 \pmod p.$$

Now use, for example, the identity

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

to get

$$\sum_{k=1}^{p-1} k^2 \equiv 0 \pmod{p}, \quad p \geq 5.$$

(e) The exponent of the largest power of p dividing $\binom{n}{m}$ is the number of carries needed to add m and $n-m$ in base p. See E. Kummer, *Jour. für Math.* **44** (1852), 115–116, and L. E. Dickson, *Quart. J. Math.* **33** (1902), 378–384.

15. (a) We have

$$1+x+x^2 = \frac{1-x^3}{1-x} \equiv (1-x)^2 \pmod{3}.$$

Hence $(1+x+x^2)^n \equiv (1-x)^{2n} \pmod{3}$. It follows easily from Exercise 1.14(a) that if $2n$ has the ternary expansion $2n = \sum a_i 3^i$, then the number of coefficients of $(1+x+x^2)^n$ not divisible by 3 is equal to $\prod(1+a_i)$. This result was obtained in collaboration with T. Amdeberhan.

(b) Let $f(n)$ be the desired number. First consider the case $n = 2^j(2^k-1)$. Since $(1+x+x^2)^{2^j} \equiv 1 + x^{2^j} + x^{2^{j+1}} \pmod{2}$, we have $f(n) = f(2^k-1)$. Now

$$(1+x+x^2)^{2^k-1} \equiv \frac{1+x^{2^k}+x^{2^{k+1}}}{1+x+x^2} \pmod{2}.$$

It is easy to check that modulo 2 we have for k odd that

$$\begin{aligned}\frac{1+x^{2^k}+x^{2^{k+1}}}{1+x+x^2} &= 1+x+x^3+x^4+x^6+x^7+\cdots+x^{2^k-2}+x^{2^k-1}+x^{2^k}\\ &\quad +x^{2^k+2}+x^{2^k+3}+x^{2^k+5}+x^{2^k+6}+\cdots+x^{2^{k+1}-3}+x^{2^{k+1}-2}.\end{aligned}$$

It follows that $f(2^k-1) = (2^{k+2}+1)/3$. Similarly, when k is even we have

$$\begin{aligned}\frac{1+x^{2^k}+x^{2^{k+1}}}{1+x+x^2} &= 1+x+x^3+x^4+x^6+x^7+\cdots+x^{2^k-4}+x^{2^k-3}+x^{2^k-1}\\ &\quad +x^{2^k+1}+x^{2^k+2}+x^{2^k+4}+x^{2^k+5}+\cdots+x^{2^{k+1}-3}+x^{2^{k+1}-2}.\end{aligned}$$

Hence in this case $f(2^k-1) = (2^{k+2}-1)/3$. For a generalization, see Exercise 4.25. Now any positive integer n can be written uniquely as $n = \sum_{i=1}^r 2^{j_i}(2^{k_i}-1)$, where $k_i \geq 1$, $j_1 \geq 0$, and $j_{i+1} > j_i + k_i$. We are simply breaking up the binary expansion of n into the maximal strings of consecutive 1's. The lengths of these strings are $k_1, \ldots, k_r$. Thus

$$(1+x+x^2)^n \equiv \prod_{i=1}^r (1+x^{2^{j_i}}+x^{2^{j_i+1}})^{2^{k_i}-1} \pmod{2}.$$

There is no cancellation among the coefficients when we expand this product since $j_{i+1} > j_i + 1$. Hence

$$f(n) = \prod_{i=1}^r f(2^{k_i}-1),$$

where $f(2^{k_i}-1)$ is given above.

Example. The binary expansion of 6039 is 1011110010111. The maximal strings of consecutive 1's have lengths 1, 4, 1 and 3. Hence

$$f(6039) = f(1)f(15)f(1)f(7) = 3 \cdot 21 \cdot 3 \cdot 11 = 2079.$$

(c) We have

$$\prod_{1\leq i<j\leq n}(x_i+x_j) \equiv \prod_{1\leq i<j\leq n}(x_i-x_j) \pmod 2,$$

where the notation means that the corresponding coefficients of each side are congruent modulo 2. The latter product is just the value of the Vandermonde determinant $\det[x_i^{j-1}]_{i,j=1}^n$, so the number of odd coefficients is $n!$. This result can also be proved by a cancellation argument; see Exercise 2.34. A more subtle result, equivalent to Exercise 4.64(a), is that the number of *nonzero* coefficients of the polynomial $\prod_{1\leq i<j\leq n}(x_i+x_j)$ is equal to the number of forests on an n-element vertex set.

Some generalizations of the results of this exercise appear in T. Amdeberhan and R. Stanley, Polynomial coefficient enumeration, preprint dated 3 February 2008;

⟨http://math.mit.edu/∼rstan/papers/coef.pdf⟩.

See also Exercise 4.24.

16. (a) This result was first given by N. Strauss as Problem 6527, *Amer. Math. Monthly* **93** (1986), 659, and later as the paper *Linear Algebra Appl.* **90** (1987), 65–72. An elegant solution to Strauss's problem was given by I. M. Gessel, *Amer. Math. Monthly* **95** (1988), 564–565, and by W. C. Waterhouse, *Linear Algebra Appl.* **105** (1988), 195–198. Namely, let V be the vector space of all functions $\mathbb{F}_p \to \mathbb{F}_p$. A basis for V consists of the functions $f_j(a) = a^j$, $0 \leq j \leq p-1$. Let $\Phi\colon V \to V$ be the linear transformation defined by $(\Phi f)(x) = (1-x)^{p-1}f(1/(1-x))$. Then it can be checked that A is just the matrix of Φ with respect to the basis f_j. It is now routine to verify that $A^3 = I$.

(b) *Answer*: $(p+2\epsilon)/3$, where $\epsilon = 1$ if $p \equiv 1 \pmod 3$ and $\epsilon = -1$ if $p \equiv -1 \pmod 3$. Both Strauss, *op. cit.*, and Waterhouse, *op. cit.*, in fact compute the Jordan normal form of A. Waterhouse uses the linear transformation Φ to give a proof similar to that given in (a).

17. (b) Think of a choice of m objects from n with repetition allowed as a placement of $n-1$ vertical bars in the slots between m dots (including slots at the beginning and end). For example,

$$|\cdot\cdot||\cdot\cdot\cdot|\cdot\cdot$$

corresponds to the multiset $\{1^0, 2^2, 3^0, 4^3, 5^2\}$. Now change the bars to dots and *vice versa*:

$$\cdot||\cdot\cdot|||\cdot||$$

yielding $\{1^1, 2^0, 3^2, 4^0, 5^0, 6^1, 7^0, 8^0\}$. This procedure gives the desired bijection. (Of course a more formal description is possible but only seems to obscure the elegance and simplicity of the above bijection.)

19. (a) One way to prove (1.120) is to recall the Lagrange interpolation formula. Namely, if $P(x)$ is a polynomial of degree less than n and $x_1, \ldots, x_n$ are distinct numbers (or indeterminates), then

$$P(x) = \sum_{i=1}^{n} P(x_i) \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

Now set $P(x) = 1$ and $x = 0$.

Applying the hint, we see that the constant term $C(a_1, \ldots, a_n)$ satisfies the recurrence

$$C(a_1, \ldots, a_n) = \sum_{i=1}^{k} C(a_1, \ldots, a_i - 1, \ldots, a_n),$$

if $a_i > 0$. If, on the other hand, $a_i = 0$, we have

$$C(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = C(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n).$$

This is also the recurrence satisfied by $\binom{a_1+\cdots+a_n}{a_1,\ldots,a_n}$, and the initial conditions $C(0, \ldots, 0) = 1$ and $\binom{0}{0,\ldots,0} = 1$ agree.

This result was conjectured by F. J. Dyson in 1962 and proved that same year by J. Gunson and K. Wilson. The elegant proof given here is due to I. J. Good in 1970. For further information and references, see [1.3, pp. 377–387].

(b) This identity is due to A. C. Dixon, Proc. London Math. Soc. **35**(1), 285–289.

(c) This is the "q-Dyson conjecture," due to G. E. Andrews, in *Theory and Application of Special Functions* (R. Askey, ed.), Academic Press, New York, 1975, pp. 191–224 (see §5). It was first proved by D. M. Bressoud and D. Zeilberger, *Discrete Math.* **54** (1985), 201–224. A more recent paper with many additional references is I. M. Gessel, L. Lv, G. Xin, and Y. Zhou, *J. Combinatorial Theory, Ser. A* **115** (2008), 1417–1435.

(d) I. G. Macdonald conjectured a generalization of (a) corresponding to any root system R. The present problem corresponds to $R = D_n$, while (a) is the case $R = A_{n-1}$ (when all the a_i's are equal). After many partial results, the conjecture was proved for all root systems by E. Opdam, *Invent. math.* **98** (1989), 1–18. Macdonald also gave a q-analogue of his conjecture, which was finally proved by I. Cherednik in 1993 and published in *Ann. Math.* **141** (1995), 191–216. For the original papers of Macdonald, see *Sem. d'Alg. Paul Dubriel et Marie-Paule Malliavin*, Lecture Notes in Math., no. 867, Springer, Berlin, pp. 90–97, and *SIAM J. Math. Anal.* **13** (1982), 988–1007.

(e) Write

$$\begin{aligned} F(x) &= F(x_1,\ldots,x_n) = \sum_{a_1,\ldots,a_n\geq 0}\left[\prod_{i=1}^n (q^{-a_i}+\cdots+q^{a_i})\right] x_1^{a_1}\cdots x_n^{a_n} \\ &= \prod_{i=1}^n \sum_{j\geq 0}(q^{-j}+\cdots+q^j)x_i^j \\ &= \prod_{i=1}^n \sum_{j\geq 0}\left(\frac{q^{-j}-q^{j+1}}{1-q}\right)x_i^j \\ &= \frac{1}{(1-q)^n}\prod_{i=1}^n\left[\frac{1}{1-q^{-1}x_i}-\frac{q}{1-qx_i}\right] \\ &= \prod_{i=1}^n \frac{1+x_i}{(1-q^{-1}x_i)(1-qx_i)}. \end{aligned}$$

We seek the term $F_0(x)$ independent from q. By the Cauchy integral formula (letting each x_i be small),

$$\begin{aligned} F_0(x) &= \frac{1}{2\pi i}\oint \frac{dq}{q}\prod_{i=1}^n \frac{1+x_i}{(1-q^{-1}x_i)(1-qx_i)} \\ &= \frac{(1+x_1)\cdots(1+x_n)}{2\pi i}\oint dq \prod_{i=1}^n \frac{q^{n-1}}{(q-x_i)(1-qx_i)}, \end{aligned}$$

where the integral is around the circle $|q|=1$. The integrand has a simple pole at $q=x_i$ with residue $x_i^{n-1}/(1-x_i^2)\prod_{j\neq i}(x_i-x_j)(1-x_ix_j)$, and the proof follows from the Residue Theorem.

NOTE. The complex analysis in the above proof can be replaced with purely formal computations using the techniques of Section 6.3.

22. (a) Let $a_1+\cdots+a_k$ be any composition of $n>1$. If $a_1=1$, then associate the composition $(a_1+a_2)+a_3+\cdots+a_k$. If $a_1>1$, then associate $1+(a_1-1)+a_2+\cdots+a_k$. This defines an involution on the set of compositions of n that changes the parity of the number of even parts. Hence the number in question is 2^{n-2}, $n\geq 2$. (Note the analogy with permutations: there are $\frac{1}{2}n!$ permutations with an even number of even cycles—namely, the elements of the alternating group.)

(b) It is easily seen that

$$\sum_{n\geq 0}(e(n)-o(n))x^n = \prod_{i\geq 1}(1+(-1)^i x^i)^{-1}.$$

In the first proof of Proposition 1.8.5 it was shown that

$$\prod_{i\geq 1}(1+x^i) = \prod_{i\geq 1}(1-x^{2i-1})^{-1}.$$

Figure 1.30: First step of the solution to Exercise 1.24

Hence (putting $-x$ for x and taking reciprocals),

$$\begin{aligned}\prod_{i\geq 1}(1+(-1)^i x^i)^{-1} &= \prod_{i\geq 1}(1+x^{2i+1})\\ &= \sum_{n\geq 0} k(n)x^n,\end{aligned}$$

by Proposition 1.8.4. A simple combinatorial proof of this exercise was given by the Cambridge Combinatorics and Coffee Club in December, 1999.

23. Form all 2^{n-1} compositions of n as in (1.19). Each bar occurs in half the compositions, so there are $(n-1)2^{n-2}$ bars in all. The total number of parts is equal to the total number of bars plus the total number of compositions, so $(n-1)2^{n-2}+2^{n-1}=(n+1)2^{n-2}$ parts in all. This argument is due to D. E. Knuth (private communication, 21 August 2007).

    Variant argument. Draw n dots in a row. Place a double bar before the first dot or in one of the $n-1$ spaces between the dots. Choose some subset of the remaining spaces between dots and place a bar in each of these spaces. The double bar and the bars partition the dots into compartments that define a composition α of n as in equation (1.19). The compartment to the right of the double bar specifies one of the parts of α. Hence the total number $f(n)$ of parts of all compositions of n is equal to the number of ways of choosing the double bar and bars as described above. As an example, the figure

    $$\cdot\,\cdot\,|\,\cdot\,||\,\cdot\,\cdot\,|\,\cdot\,\cdot\,\cdot$$

    corresponds to the composition $(2,1,2,3)$ of 8 with the third part selected.

    If we place the double bar before the first dot, then there are 2^{n-1} choices for the remaining bars. Otherwise there are $n-1$ choices for the double bar and then 2^{n-2} choices for the remaining bars. Hence $f(n)=2^{n-1}+(n-1)2^{n-2}=(n+1)2^{n-2}$.

24. Draw a line of n dots and circle k consecutive dots. Put a vertical bar to the left and right of the circled dots. For example, $n=9$, $k=3$: see Figure 1.30.

    Case 1. The circled dots don't include an endpoint. The above procedure can then be done in $n-k-1$ ways. Then there remain $n-k-2$ spaces between uncircled dots. Insert at most one vertical bar in each space in 2^{n-k-2} ways. This defines a composition with one part equal to k circled. For example, if we insert bars as in Figure 1.31 then we obtain $3+1+1+③+1$.

    Case 2. The circled dots include an endpoint. This happens in two ways, and now there are $n-k-1$ spaces into which bars can be inserted in 2^{n-k-1} ways.

    Hence we get the answer

    $$(n-k-1)2^{n-k-2}+2\cdot 2^{n-k-1}=(n-k+3)2^{n-k-2}.$$

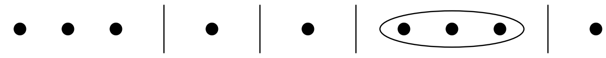


Figure 1.31: Continuation of the solution to Exercise 1.24

25. It is clear that

$$\sum_{n,r,s} f(n,r,s)q^r t^s x^n = \sum_{j\geq 0}\left(\frac{qx}{1-x^2}+\frac{tx^2}{1-x^2}\right)^j.$$

The coefficient of $q^r t^s$ is given by

$$\binom{r+s}{r}\frac{x^{r+2s}}{(1-x^2)^{r+s}} = \binom{r+s}{r}\sum_{m\geq 0}\binom{m+r+s-1}{r+s-1}x^{2m+r+2s},$$

and the proof follows.

For a bijective proof, choose a composition of $r+k$ into $r+s$ parts in $\binom{r+k-1}{r+s-1}$ ways. Multiply r of these parts by 2 in $\binom{r+s}{r}$ ways. Multiply each of the other parts by 2 and subtract 1. We obtain each composition of n with r odd parts and s even parts exactly once, and the proof follows.

27. *Answer:* $(n+3)2^{n-2}-1$.

30. Let $b_i = a_i - i + 1$. Then $1 \leq b_1 \leq b_2 \leq \cdots \leq b_k \leq n-k+1$ and each b_i is odd. Conversely, given the b_i's we can uniquely recover the a_i's. Hence setting $m = \lfloor (n-k+2)/2 \rfloor$, the number of odd integers in the set $[n-k+1]$, we obtain the answer $\left(\!\binom{m}{k}\!\right) = \binom{m+k-1}{k} = \binom{q}{k}$, where $q = \lfloor (n+k)/2 \rfloor$.

This exercise is called *Terquem's problem.* For some generalizations, see M. Abramson and W. O. J. Moser, *J. Combinatorial Theory* **7** (1969), 171–180; S. M. Tanny, *Canad. Math. Bull.* **18** (1975), 769–770; J. de Biasi, *C. R. Acad. Sci. Paris Sér. A-B* **285** (1977), A89–A92; and I. P. Goulden and D. M. Jackson, *Discrete Math.* **22** (1978), 99–104. A further generalization is given by Exercise 1.10.

31. (a) $x(x+1)(x+2)\cdots(x+n-1) = n!\left(\!\binom{n+1}{x-1}\!\right) = n!\left(\!\binom{x}{n}\!\right)$

(b) $(n)_x(n-1)_{n-x} = n!\binom{n-1}{x-1}$

(c) $\sum_{k=1}^{x}\frac{n!}{k!}\binom{n-1}{k-1}$

32. The key feature of this problem is that each element of S can be treated *independently*, as in Example 1.1.16.

(a) For each $x \in S$, we may specify the least i (if any) for which $x \in T_i$. There are $k+1$ choices for each x, so $(k+1)^n$ ways in all.

Figure 1.32: An illustration of the solution to Exercise 1.35(f)

(b) Now each x can be in at most one T_i, so again there are $k+1$ choices for x and $(k+1)^n$ choices in all. (In fact, there is a very simple bijection between the sequences enumerated by (a) and (b).)

(c) Now each x can be in any subset of the T_i's except the subset $\emptyset$. Hence there are $2^k - 1$ choices for each x and $(2^k-1)^n$ ways in all.

34. Let $b_i = a_i - (i-1)j$ to get $1 \le b_1 \le \cdots \le b_k \le n-(k-1)j$, so the number of sequences is $\left(\!\binom{n-(k-1)j}{k}\!\right)$.

35. (a) Obtain a recurrence by considering those subsets S which do or do not contain n. *Answer:* F_{n+2}.

(b) Consider whether the first part is 2 or at least 3. *Answer:* F_{n-1}.

(c) Consider whether the first part is 1 or 2. *Answer:* F_{n+1}.

(d) Consider whether the first part is 1 or at least 3. *Answer:* F_n.

(e) Consider whether $\varepsilon = 0$ or 1. *Answer:* F_{n+2}.

(f) The following proof, as well as the proofs of (g) and (h), are due to Ira Gessel. Gessel (private communication, 2 May 2007) has developed a systematic approach to "Fibonacci composition formulas" based on factorization in free monoids as discussed in Section 4.7. The sum $\sum a_1 a_2 \cdots a_k$ counts the number of ways of inserting at most one vertical bar in each of the $n-1$ spaces separating a line of n dots, and then circling one dot in each compartment. An example is shown in Figure 1.32. Replace each bar by a 1, each uncircled dot by a 2, and each circled dot by a 1. For example, Figure 1.32 becomes

2 1 2 2 1 1 2 1 1 1 1 1 2 1 1 1 2 2 1 1 1 2 1 2.

We get a composition of $2n-1$ into 1's and 2's, and this correspondence is invertible. Hence by (c) the answer is F_{2n}.

A simple generating function proof can also be given using the identity

$$\begin{aligned}\sum_{k\ge1}(x+2x^2+3x^3+\cdots)^k &= \frac{x/(1-x)^2}{1-x/(1-x)^2}\\ &= \frac{x}{1-3x+x^2}\\ &= \sum_{n\ge1} F_{2n}x^n.\end{aligned}$$

(g) Given a composition $(a_1,\ldots,a_k)$ of n, replace each part a_i with a composition α_i of $2a_i$ into parts 1 and 2, such that α_i begins with a 1, ends in a 2, and for all j the $2j$-th 1 in α is followed by a 1, unless this $2j$-th 1 is the last 1 in α. For instance, the part $a_i = 4$ can be replaced by any of the seven compositions 1111112, 111122, 111212, 11222, 121112, 12122, 12212. It can be checked that (i) every composition of $2n$ into parts 1 and 2, beginning with 1 and ending with 2, occurs exactly once by applying this procedure to all compositions of n, and (ii) the number of compositions that can replace a_i is $2^{a_i-1}-1$. It follows from part (c) that the answer is F_{2n-2}. A generating function proof takes the form

$$\begin{aligned}\sum_{k\geq 1}(x^2+3x^3+7x^4+\cdots)^k &= \frac{x^2/(1-x)(1-2x)}{1-x^2/(1-x)(1-2x)}\\ &= \frac{x^2}{1-3x+x^2}\\ &= \sum_{n\geq 2}F_{2n-2}x^n.\end{aligned}$$

(h) Given a composition $(a_1,\ldots,a_k)$ of n, replace each 1 with either 2 or $1,1$, and replace each $j>1$ with $1,2,\ldots,2,1$, where there are $j-1$ 2's. Every composition of $2n$ with parts 1 and 2 is obtained in this way, so from part (c) we obtain the answer F_{2n+1}. A generating function proof takes the form

$$\begin{aligned}\frac{1}{1-2x-x^2-x^3-x^4-\cdots} &= \frac{1}{1-x-\frac{x}{1-x}}\\ &= \frac{1-x}{1-3x+x^2}\\ &= \sum_{n\geq 0}F_{2n+1}x^n.\end{aligned}$$

(i) *Answer:* $2F_{3n-4}$ (with F_n defined for *all* $n\in\mathbb{Z}$ using the recurrence $F_n = F_{n-1}+F_{n-2}$), a consequence of the expansion

$$\frac{1}{1+\frac{x}{1-5x}+\frac{x}{1-x}} = 1-2\sum_{n\geq 1}F_{3n-4}x^n.$$

A bijective proof is not known. This result is due to D. E. Knuth (private communication, 21 August 2007).

(k) *Answer:* F_{2n+2}. Let $f(n)$ be the number in question. Now

$$P_n = P_{n-1}+P_{n-1}x_n+P_nx_{n+1}. \tag{1.153}$$

Each term of the above sum has $f(n-1)$ terms when expanded as a polynomial in the x_i's. Since

$$P_{n-1}+P_{n-1}x_n = P_{n-2}(1+x_{n-1}+x_n)+P_{n-2}(1+x_{n-1}+x_n)x_n,$$

the only overlap between the three terms in equation (1.153) comes from $P_{n-2}x_n$, which has $f(n-2)$ terms. Hence $f(n)=3f(n-1)-f(n-2)$, from which the proof follows easily. This problem was derived from a conjecture of T. Amdeberhan (November 2007). For a variant, see Exercise 4.20.

36. Let $f_n(k)$ denote the answer. For each $i \in [n]$ we can decide which T_j contains i independently of the other $i' \in [n]$. Hence $f_n(k) = f_k(1)^n$. But computing $f_k(1)$ is equivalent to Exercise 1.35(e). Hence $f_n(k) = F_{k+2}^n$.

37. While it is not difficult to show that the right-hand side of equation (1.122) satisfies the Fibonacci recurrence and initial conditions, we prefer a more combinatorial proof. For instance, Exercise 1.34 in the case $j = 2$ shows that $\binom{n-k}{k}$ is the number of k-subsets of $[n-1]$ containing no two consecutive integers. Now use Exercise 1.35(a).

39. *First solution* (sketch). Let $a_{m,n}$ be the number of ordered pairs (S, T) with $S \subseteq [m]$ and $T \subseteq [n]$ satisfying $s > \#T$ for all $s \in S$ and $t > \#S$ for all $t \in T$. An easy bijection gives

$$a_{m,n} = a_{m-1,n} + a_{m-1,n-1}.$$

Using $a_{ij} = a_{ji}$ we get

$$\begin{aligned} a_{n,n} &= a_{n,n-1} + a_{n-1,n-1} \\ a_{n,n-1} &= a_{n-1,n-1} + a_{n-1,n-2}, \end{aligned}$$

from which it follows (using the initial conditions $a_{0,0} = 1$ and $a_{1,0} = 2$) that $a_{n,n} = F_{2n+2}$ and $a_{n,n-1} = F_{2n+1}$.

Second solution (sketch). It is easy to see that

$$a_{m,n} = \sum_{\substack{i,j \geq 0 \\ i+j \leq \min\{m,n\}}} \binom{m-j}{i}\binom{n-i}{j}.$$

It can then be proved bijectively that $\sum_{\substack{i,j\geq 0 \\ i+j\leq n}} \binom{n-j}{i}\binom{n-i}{j}$ is the number of compositions of $2n+1$ with parts 1 and 2. The proof follows from Exercise 1.35(c).

This problem (for the case $n = 10$) appeared as Problem A-6 on the Fifty-First William Lowell Putnam Mathematical Competition (1990). The two solutions above appear in K. S. Kedlaya, B. Poonen, and R. Vakil, *The William Lowell Putnam Mathematical Competition*, Mathematical Association of America, Washington, DC, 2002 (pp. 123–124).

41. (a) Perhaps the most straightforward solution is to let $\#S = k$, giving

$$\begin{aligned} f(n) &= \sum_{k=0}^{n} (n-k)_k (n-k)! \binom{n}{k} \\ &= n! \sum_{k=0}^{n} \binom{n-k}{k}. \end{aligned}$$

Now use Exercise 1.37. It is considerably trickier to give a direct bijective proof.

(b) We now have

$$\begin{aligned} g(n) &= \sum_{k=0}^{n-1} (n-k)_k (n-k-1)! \binom{n}{k} \\ &= (n-1)! \sum_{k=0}^{n-1} \frac{n}{n-k} \binom{n-k}{k}. \end{aligned}$$

There are a number ways to show that $L_n = \sum_{k=0}^{n-1} \frac{n}{n-k}\binom{n-k}{k}$, and the proof follows. This result was suggested by D. E. Knuth (private communication, 21 August 2007) upon seeing (a). A simple bijective proof was suggested by R. X. Du (private communication, 27 March 2011); namely, choose an n-cycle C in $(n-1)!$ ways, and regard the elements of C as n points on a circle. We can choose S to be any subset of the points, no two consecutive. By Exercise 1.40 this can be done in L_n ways, so the proof follows.

42. Let $\prod_{n\geq 2}(1-x^{F_n}) = \sum_{k\geq 0} a_k x^k$. Split the interval $[F_n, F_{n+1}-1]$ into the three subintervals $[F_n, F_n+F_{n-3}-2]$, $[F_n+F_{n-3}-1, F_n+F_{n-2}-1]$, and $[F_n+F_{n-2}, F_{n+1}-1]$. The following results can be shown by induction:

- The numbers $a_{F_n}, a_{F_n+1}, \dots, a_{F_n+F_{n-3}-2}$ are equal to the numbers $(-1)^{n-1}a_{F_{n-3}-2}$, $(-1)^{n-1}a_{F_{n-3}-3}, \dots, (-1)^{n-1}a_0$ in that order.
- The numbers $a_{F_n+F_{n-3}-1}$, $a_{F_n+F_{n-3}}, \dots, a_{F_n+F_{n-2}-1}$ are equal to 0.
- The numbers $a_{F_n+F_{n-2}}$, $a_{F_n+F_{n-2}+1}, \dots, a_{F_{n+1}-1}$ are equal to the numbers a_0, a_1, $\dots$, $a_{F_{n-3}-1}$ in that order.

From these results the proof follows by induction.

N. Robbins, *Fibonacci Quart.* **34.4** (1996), 306-313, was the first to prove that the coefficients are $0, \pm 1$. The above explicit recursive description of the coefficents is due to F. Ardila, *Fibonacci Quart.* **42** (2004), 202–204. Another elegant proof was later given by Y. Zhao, The coefficients of a truncated Fibonacci series, *Fib. Quarterly*, to appear, and a significant generalization by H. Diao, `arXiv:0802.1293`.

43. *Answer:*

$$\begin{aligned} S(n,1) &= 1 & c(n,1) &= (n-1)! \\ S(n,2) &= 2^{n-1}-1 & c(n,2) &= (n-1)!H_{n-1} \\ S(n,n) &= 1 & c(n,n) &= 1 \\ S(n,n-1) &= \binom{n}{2} & c(n,n-1) &= \binom{n}{2} \\ S(n,n-2) &= \binom{n}{3} + 3\binom{n}{4} & c(n,n-2) &= 2\binom{n}{3} + 3\binom{n}{4}. \end{aligned}$$

An elegant method for computing $c(n,2)$ is the following. Choose a permutation $a_1a_2\cdots a_n \in \mathfrak{S}_n$ with $a_1 = 1$ in $(n-1)!$ ways. Choose $1 \leq j \leq n-1$ and let

w be the permutation whose disjoint cycle form is $(a_1, a_2, \ldots, a_j)(a_{j+1}, a_{j+2}, \ldots, a_n)$. We obtain exactly j times every permutation with two cycles such that the cycle not containing 1 has length $n-j$. Hence $c(n,2) = (n-1)!H_{n-1}$.

As a further example, let us compute $S(n, n-2)$. The block sizes of a partition of $[n]$ with $n-2$ blocks are either 3 (once) and 1 ($n-3$ times), or 2 (twice) and 1 ($n-4$ times). In the first case there are $\binom{n}{3}$ ways of choosing the 3-element block. In the second case there are $\binom{n}{4}$ ways of choosing the union of the two 2-element blocks, and then three ways to choose the blocks themselves. Hence $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ as claimed.

45. Define $a_{i+1} + a_{i+2} + \cdots + a_k$ to be the least r such that when $1, 2, \ldots, r$ are removed from π, the resulting partition has i blocks.

46. (a) We have by equation (1.94c) that

$$\begin{aligned} \sum_{n\geq 0} S(n,k)x^n &= \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \\ &= \frac{x^k}{(1-x)^{\lceil k/2 \rceil} \pmod 2}. \end{aligned}$$

(b) The first of several persons to find a combinatorial proof were K. L. Collins and M. Hovey, *Combinatorica* **31** (1991), 31–32. For further congruence properties of $S(n,k)$, see L. Carlitz, *Acta Arith.* **10** (1965), 409–422.

(c) Taking equation (1.28) modulo 2 gives

$$\sum_{k=0}^{n} c(n,k)t^k = t^{\lceil n/2 \rceil}(t+1)^{\lfloor n/2 \rfloor} \pmod 2.$$

Hence

$$c(n,k) \equiv \binom{\lfloor n/2 \rfloor}{k - \lceil n/2 \rceil} = \binom{\lfloor n/2 \rfloor}{n-k} \pmod 2.$$

48. (a) This remarkable result is due to J. N. Darroch, *Ann. Math. Stat.* **35** (1964), 1317–1321. For a nice exposition including much related work, see J. Pitman, *J. Combinatorial Theory, Ser. A* **77** (1997), 279–303.

(b) Let $P(x) = \sum_{k=0}^{n} c(n,k)x^k$. It is routine to compute from Proposition 1.3.7 that

$$\frac{P'(1)}{P(1)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

and the proof follows from (a). For further information on the distribution of the number of cycles of a permutations $w \in \mathfrak{S}_n$, see Pitman, *ibid.*, pp. 289–290.

(c) This result is due to E. R. Canfield and C. Pomerance, *Integers* **2** (2002), A1 (electronic); *Corrigendum* **5**(1) (2005), A9, improving earlier expressions for K_n due to Canfield and Menon (independently). Previously it was shown by L. H.

Harper, *Ann. Math. Stat.* **38** (1966), 410–414 (Lemma 1), that the polynomial $\sum_k S(n,k)x^k$ has real zeros. As Pitman points out in his paper cited above (page 291), the result (a) of Darroch reduces the problem of estimating K_n to estimating the expected number of blocks of a random partition of $[n]$. For further discussion, see D. E. Knuth, *The Art of Computer Programming*, vol. 4, Fascicle 3, Addison-Wesley, Upper Saddle River, NJ, 2005 (Exercises 7.2.1.5–62 and 7.2.1.5–63(e)).

49. (a) Let $F_d(x) = A_d(x)/(1-x)^{d+1}$. Differentiate equation (1.37) and multiply by x, yielding

$$F_{d+1}(x) = x\frac{d}{dx}F_d(x),$$

etc.

(b) The proof is by induction on d. Since $A_1(x) = x$, the assertion is true for $d = 1$. Assume the assertion for d. By Rolle's theorem, the function $f(x) = \frac{d}{dx}(1-x)^{-d-1}A_d(x)$ has $d-1$ simple negative real zeros that interlace the zeros of $A_d(x)$. Since $\lim_{x\to-\infty} f(x) = 0$, there is an additional zero of $f(x)$ less than the smallest zero of $A_d(x)$. Using equation (1.38), we have accounted for d strictly negative simple zeros of $A_{d+1}(x)$, and $x = 0$ is an additional zero. The proof follows by induction. This result can be extended to permutations of a multiset; see R. Simion, *J. Combinatorial Theory, Ser. A* **36** (1984), 15–22.

50. (b) Let $D = d/dx$. By Rolle's theorem, $Q(x) = D^{i-1}P(x)$ has real zeros, and thus also $R(x) = x^{n-i+1}Q(1/x)$. Again by Rolle's theorem, $D^{n-i-1}R(x)$ has real zeros. But one computes easily that

$$D^{n-i-1}R(x) = \frac{n!}{2}\left(b_{i-1}x^2 + 2b_ix + b_{i+1}\right).$$

In order for this quadratic polynomial to have real zeros, we must have $b_i^2 \geq b_{i-1}b_{i+1}$. This result goes back to I. Newton; see e.g. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, second ed., Cambridge University Press, Cambridge, England, 1952 (page 52).

(c) Let us say that a polynomial $P(x) = \sum_{i=0}^m a_ix^i$ with coefficients satisfying $a_i = a_{m-i}$ has *center* $m/2$. (We don't assume that $\deg P(x) = m$, i.e., we may have $a_m = 0$.) Thus $P(x)$ has center $m/2$ if and only if $P(x) = x^mP(1/x)$. If also $Q(x) = x^nQ(1/x)$ (so $Q(x)$ has center $n/2$), then $P(x)Q(x) = x^{m+n}P(1/x)Q(1/x)$. Thus $P(x)Q(x)$ has symmetric coefficients (with center $(m+n)/2$). It is also easy to show this simply by computing the coefficients of $P(x)Q(x)$ in terms of the coefficients of $P(x)$ and $Q(x)$.

Now assume that $P(x) = \sum_{i=0}^m a_ix^i$ has center $m/2$ and has unimodal coefficients, and similarly for $Q(x) = \sum_{i=0}^n b_ix^i$. Let $A_j(x) = x^j + x^{j+1} + \cdots + x^{m-j}$, a polynomial with center $m/2$, and similarly $B_j(x) = x^j + x^{j+1} + \cdots + x^{n-j}$. It is easy to see that

$$\begin{aligned} P(x) &= \sum_{i=0}^{\lfloor m/2 \rfloor} (a_i - a_{i-1}) A_i(x) \\ Q(x) &= \sum_{j=0}^{\lfloor n/2 \rfloor} (b_j - b_{j-1}) B_j(x). \end{aligned}$$

Thus

$$P(x)Q(x) = \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} (a_i - a_{i-1})(b_j - b_{j-1}) A_i(x) B_j(x).$$

It is easy to check by explicit computation that $A_i(x)B_j(x)$ has unimodal coefficients and center $(m+n)/2$. Since $P(x)$ and $Q(x)$ have unimodal coefficients, we have

$$(a_i - a_{i-1})(b_j - b_{j-1}) \geq 0.$$

Hence we have expressed $P(x)Q(x)$ as a nonnegative linear combination of unimodal polynomials, all with the same center $(m+n)/2$. It follows that $P(x)Q(x)$ is also unimodal (with center $(m+n)/2$).

(d) Perhaps the most elegant proof (and one suggesting some nice generalizations) uses linear algebra. Write $P(x) = \sum_{i=0}^{m} a_i x^i$ and $Q(x) = \sum_{i=0}^{n} b_i x^i$. Set $a_i = 0$ if $i \notin [0, m]$, and similarly for b_i. If X and Y are $r \times r$ real matrices all of whose $k \times k$ minors are nonnegative, then the Cauchy-Binet theorem shows that the same is true for the matrix XY. Moreover, it is easily seen that if $c_0, c_1, \ldots, c_n$ is nonnegative and log-concave with no internal zeros, then $c_i c_j \geq c_{i-s} c_{j+s}$ whenever $i \leq j$ and $s \geq 0$. Now take $k = 2$, $X = [a_{j-i}]_{i,j=0}^{m+n}$, and $Y = [b_{j-i}]_{i,j=0}^{m+n}$, and the proof follows.

(e) The symmetry of the two polynomials is easy to see in various ways. The polynomial $x \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)}$ is the Eulerian polynomial $A_n(x)$ by equation (1.36); now use (a), (b) and Exercise 1.49. The unimodality of the polynomial $\sum_{w \in \mathfrak{S}_n} x^{\mathrm{inv}(w)}$ follows from (c) and the product formula (1.30). NOTE. A combinatorial proof of the unimodality of $\sum_{w \in \mathfrak{S}_n} x^{\mathrm{inv}(w)}$ is implicit in the proof we have given, while a combinatorial proof of the log-concavity and unimodality of $A_n(x)$ is due to V. Gasharov, *J. Combinatorial Theory, Ser. A* **82** (1998), 134–146 (§§4–5).

(f) This result was proved by F. De Mari and M. Shayman, *Acta Appl. Math.* **12** (1988), 213–235, using the hard Lefschetz theorem from algebraic geometry. It would be interesting to give a more elementary proof. A related result was proved by M. Bóna, Generalized descents and normality, `arXiv:0709.4483`.

(g) Let $n = 4$ and $S = \{(1,2), (2,3), (3,4), (1,4)\}$. Then

$$P_S(x) = x^4 + 8x^3 + 6x^2 + 8x + 1.$$

Note that part (f) asserts that $P_S(x)$ is unimodal for $S = \{(i,j) : 1 \leq i < j \leq n,\ j \leq i + p\}$. It seems likely (though this has not been checked) that the proof of De Mari and Shayman can be extended to the case $S = \{(i,j) : 1 \le i < j \le n,\ j \le i + p_i\}$, where $p_1, \dots, p_{n-1}$ are any nonnegative integers. Can anything further be said about those S for which $P_S(x)$ is unimodal?

For further information on the fascinating topic of unimodal and log-concave sequences, see R. Stanley, in *Graph Theory and Its Applications: East and West*, Ann. New York Acad. Sci., vol. 576, 1989, pp. 500–535, and the sequel by F. Brenti, in *Contemp. Math.* **178**, Amer. Math. Soc., Providence, RI, 1994, pp. 71–89. For the unimodality of the q-binomial coefficient $\binom{\boldsymbol{n}}{\boldsymbol{k}}$ and related results, see Exercise 7.75.

51. This result goes back to P. S. de Laplace. The following proof is due to R. Stanley, in *Higher Combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976)*, M. Aigner, ed., Reidel, Dordrecht/Boston, 1977, p. 49. Given $w \in \mathfrak{S}_n$, let $\mathcal{S}_w$ denote the region (a simplex) in $\mathbb{R}^n$ defined by

$$0 \le x_{w(1)} \le x_{w(2)} \le \cdots \le x_{w(n)} \le 1.$$

Define $\mathcal{S}_{nk} = \bigcup_w \mathcal{S}_w$, where w ranges over all permutations in $\mathfrak{S}_n$ with exactly $k-1$ descents. It is easy to see that $\mathrm{vol}(\mathcal{S}_w) = 1/n!$, so $\mathrm{vol}(\mathcal{S}_{nk}) = A(n,k)/n!$. Define a map $\varphi\colon \mathcal{S}_{nk} \to \mathcal{R}_{nk}$ by $\varphi(x_1, \dots, x_n) = (y_1, \dots, y_n)$, where

$$y_i = \begin{cases} x_{i+1} - x_i, & \text{if } x_i < x_{i+1} \\ 1 + x_{i+1} - x_i, & \text{if } x_i > x_{i+1}. \end{cases}$$

Here we set $x_{n+1} = 1$, and we leave φ undefined on the set of measure zero consisting of points where some $x_{i-1} = x_i$. One can check that φ is measure-preserving and a bijection up to a set of measure zero. Hence $\mathrm{vol}(\mathcal{R}_{nk}) = \mathrm{vol}(\mathcal{S}_{nk}) = A(n,k)/n!$. For some additional proofs, see W. Meyer and R. von Randow, *Math. Annalen* **193** (1971), 315–321, and S. M. Tanny, *Duke Math. J.* **40** (1973), 717–722, and J. W. Pitman, *J. Combinatorial Theory, Ser. A* **77** (1997), 279–303 (pp. 295–296). For a refinement and further references, see R. Ehrenborg, M. A. Readdy, and E. Steingrímsson, *J. Combinatorial Theory, Ser. A* **81** (1998), 121–126. For some related results, see Exercise 4.62.

52. This amusing result is due to J. Holte, *Amer. Math. Monthly* **104** (1997), 138–149. Holte derived this result in the setting of Markov chains and obtained many additional results about the combinatorics of carrying. Further work on this subject is due to P. Diaconis and J. Fulman, *Amer. Math. Monthly* **116** (2009), 788–803, and *Advances in Applied Math.* **43** (2009), 176–196, and A. Borodin, P. Diaconis, and J. Fulman, *Bull Amer. Math. Soc.* **47** (2009), 639–670. There is a simple intuitive reason, which is not difficult to make rigorous, why we get the Eulerian numbers. The probability that we carry j in a certain column is roughly the probability that if $i_1, \dots, i_n$ are random integers in the interval $[0, b-1]$, then $bj \le i_1 + \cdots + i_n < b(j+1)$. Now divide by b and use Exercise 1.51.

56. Let $\phi(w)$ denote the standardization (as defined in the second proof of Proposition 1.7.1) of $w \in \mathfrak{S}_M$. If $M = \{1^{m_1}, 2^{m_2}, \dots\}$ and $\#M = n$, then $\{\phi(w) : w \in \mathfrak{S}_M\}$ consists of all permutations $v \in \mathfrak{S}_n$ such that $D(v^{-1}) = \{m_1, m_1 + m_2, \cdots\} \cap [n-1]$. It is

easy to see that $\text{inv}(w) = \text{inv}(v)$ (a special case of (1.71)) and $\text{maj}(w) = \text{maj}(v)$. The proof now follows from equation (1.43) and Theorem 1.4.8. This result is due to P. A. MacMahon, stated explicitly on page 317 of his paper [1.54]. Some other classes of permutations that are equidistributed with respect to inv and maj are given by A. Björner and M. L. Wachs, *J. Combinatorial Theory, Ser. A* **52** (1989), 165–187, and D. Foata and D. Zeilberger, *J. Comput. Applied Math.* **68** (1996), 79–101. See also the solution to Exercise 5.49(e).

57. Condition (i) does not hold if and only if there are indices $i < i'$ and $j < j'$ such that $(i, j) \in D(w)$, $(i', j') \in D(w)$, $(i, j') \notin D(w)$, $(i', j) \notin D(w)$. Let $w(i'') = j$ and $w(i''') = j'$. It is easy to check by drawing a diagram that $i < i'' < i' < i'''$ and $w(i'') < w(i) < w(i''') < w(i')$, so w is not 2143-avoiding. The steps are reversible, so (i) and (iii) are equivalent. The equivalence of (i) and (ii) follows from the fact that the jth term of $I(w)$ (respectively, $I(w^{-1})$) is the number of elements of $D(w)$ in column (respectively, row) j.

    The permutations of this exercise are called *vexillary.* For further information on their history and properties, see Exercise 7.22(d,e).

58. (b) The final step in obtaining this result was achieved by Z. Stankova, *Europ. J. Combin.* **17** (1996), 501–517. For further information, see H. S. Wilf, *Discrete Math.* **257** (2002), 575–583, and M. Bóna [1.11, §4.4].

59. This result is known as the *Stanley-Wilf conjecture.* It was shown by R. Arratia, *Electronic J. Combinatorics* **6**(1) (1999), N1, that the conjecture follows from the statement that there is a real number $c > 1$ (depending on u) for which $s_u(n) < c^n$ for all $n \geq 1$. This statement was given a surprisingly simple and elegant proof by A. Marcus and G. Tardos, *J. Combinatorial Theory, Ser. A* **107** (2004), 153–160. A nice exposition of this proof due to D. Zeilberger is available at

    ⟨`www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/paramath.html`⟩.

    Another nice exposition is given by M. Bóna, [1.11, §4.5].

60. *Answer.* The equivalence classes consist of permutations whose inverses have a fixed descent set. The number of equivalence classes is therefore 2^{n-1}, the number of subsets of $[n-1]$.

    While it is not difficult to prove this result directly, it also can be understood in a nice way using the "Cartier-Foata theory" of Exercise 3.123.

61. (a) By the properties of the bijection $w \mapsto T(w)$ discussed in Section 1.5, we have that

$$F(x; a, b, c, d) = \sum_{n \geq 1} \sum_{T} a^{\text{lr}(T)} b^{e(T)-1} c^{r(T)} d^{l(T)} \frac{x^n}{n!},$$

where T ranges over all increasing binary trees on the vertex set $[n]$, with $\text{lr}(T)$ vertices with two children, $e(T)$ vertices that are endpoints, $l(T)$ vertices with just a left child, and $r(T)$ vertices with just a right child. By removing the root from T, we obtain the equation

$$\frac{\partial}{\partial x}(F - bx) = abF^2 + (c+d)F. \tag{1.154}$$

Solving this equation (a Ricatti equation, with a well-known method of solution) with the initial condition $F(0; a, b, c, d) = 0$ yields equation (1.124).

This result is due to L. Carlitz and R. Scoville, *J. reine angew. Math.* **265** (1974), 110–137 (§7). Our presentation follows Exercise 3.3.46 of I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration*, John Wiley & Sons, New York, 1983; reprinted by Dover, Mineola, NY, 2004. This latter reference contains more details on solving the differential equation (1.154).

(b) The generating function is given by $1 + tF(x; 1, t, 1, 1)$, which can be simplified to the right-hand side of equation (1.125).

The enumeration of permutations by number of peaks was first considered by F. N. David and D. E. Barton, *Combinatorial Chance*, Hafner, New York, 1962 (pp. 162–164). They obtain a generating function for $r(n, k)$ written in a different form from equation (1.125).

(c) We have that $f(n)$ is the number of increasing binary trees on $[n]$ such that no vertex has only a left child except possibly the last vertex obtained by beginning with the root and taking right children. Let $g(n)$ be the number of increasing binary trees on $[n]$ such that no vertex has only a left child. Then

$$\begin{aligned} f(n+1) &= \sum_{k=0}^{n} \binom{n}{k} f(k) g(n-k) \\ g(n+1) &+ \sum_{k=0}^{n-1} \binom{n}{k} g(k) g(n-k), \end{aligned}$$

with $f(0) = g(0) = 1$. Setting $F(x) = \sum f(n)x^n/n!$ and $G(x) = \sum g(n)x^n/n!$, we obtain $F' = FG$ and $G + G' = G^2 + 1$. We can solve these differential equations to obtain equation (1.126). Goulden and Jackson, *op. cit.* (Exercise 5.2.17, attribution on page 306) attribute this result to P. Flajolet (private communication, 1982). The proof in Goulden and Jackson is based essentially on the Principle of Inclusion-Exclusion, and is given here in Exercise 2.23.

62. (b) First note that

$$p_k^n = \sum_{d|n} \sum_{A} d(w(A))^{nk/d}, \tag{1.155}$$

where A ranges over all aperiodic cycles of length d (i.e., cycles of length d that are unequal to a proper cyclic shift of themselves). Now substitute (1.155) into the expansion of $\log \prod (1 - p_k)^{-1}$ and simplify.

This result is implicit in the work of R. C. Lyndon (see Lothaire [4.31, Thm. 5.1.5]). See also N. G. de Bruijn and D. A. Klarner, *SIAM J. Alg. Disc. Meth.* **3** (1982), 359–368. The result was stated explicitly by I. M. Gessel (unpublished). A different theory of cycles of multiset permutations, due to D. Foata, has a nice exposition in §5.1.2 of D. E. Knuth [1.48]. In Foata's theory, a multiset permutation has the meaning of Section 1.7.

(c) Let $x_1 = \cdots = x_k = x$, and $x_j = 0$ if $j > k$.

(d) Let $\sigma = (a_1, a_2, \ldots, a_{jk})$ be a multiset cycle of length jk, where k is the largest integer for which the word $u = a_1 a_2 \cdots a_{jk}$ has the form v^k for some word v of length j (where v^k denotes the concatenation of k copies of v). Let $\Gamma(\sigma) = p_k^j$. Given a multiset permutation $\pi = \sigma_1\sigma_2\cdots\sigma_m$ where each σ_i is a multiset cycle, define $\Gamma(\pi) = \Gamma(\sigma_1)\cdots\Gamma(\sigma_m)$. It can then be verified combinatorially that the number of multiset permutations π with fixed $w(\pi)$ and $\Gamma(\pi)$ is equal to the coefficient of $w(\pi)$ in $\Gamma(\pi)$, leading to the desired bijection.

63. Label the envelopes $1, 2, \ldots, n$ in decreasing order of size. Partially order an arrangement of envelopes by inclusion, and adjoin a root labelled 0 at the top. We obtain an (unordered) increasing tree on $n+1$ vertices, and this correspondence is clearly invertible. Hence by Proposition 1.5.5 there are $n!$ arrangements in all, of which $c(n,k)$ have k envelopes not contained in another and $A(n,k)$ have k envelopes not containing another.

64. (a) Let u be a sequence being counted, with m_i occurrences of i. Replace the 1's in u from right-to-left by $1, 2, \ldots, m_1$. Then replace the 2's from right-to-left by $m_1+1, m_1+2, \ldots, m_1+m_2$, etc. This procedure gives a bijection with $\mathfrak{S}_n$. For instance, 13213312 corresponds to 38527614. Note that this bijection could also be described as $u \mapsto \rho\psi\rho(u)$, where $\rho(v)$ is the reversal of v, and ψ denotes standardization (defined after the second proof of Proposition 1.7.1).

(b) The bijection in (a) has the property that $\max\{a_1, \ldots, a_n\} = \operatorname{des}(\rho(w)^{-1})+1$, etc. This result was pointed out by D. E. Knuth (private communication, 21 August 2007) upon seeing (a).

65. It follows from a general theorem of Ramanujan (see D. Zagier, in J. H. Bruinier, G. van der Geer, G. Harder and D. Zagier, eds., *The 1-2-3 of Modular Forms*, Springer-Verlag, Berlin, 2008 (Prop. 16, p. 49)) that y satisfies a third order algebraic differential equation, but it is considerably more complicated than the fourth degree equation (1.127). This equation was first computed by M. Rubey in 2010. See W. Hebisch and M. Rubey, *J. Symbolic Computation*, to appear.

70. (a) Draw a line L along the main diagonal of the Ferrers diagram of λ. Then a_i is the number of dots in the ith row to the right of L, while b_i is the number of dots in the ith column below i. Figure 1.33 shows that $A_{77421} = \begin{pmatrix} 6 & 5 & 1 \\ 4 & 2 & 0 \end{pmatrix}$. This bijection is due to F. G. Frobenius, *Sitz. Preuss. Akad. Berlin* (1900), 516–534, and *Gesammelte Abh.* **3**, Springer, Berlin, 1969, pp. 148–166, and the array A_λ is called the *Frobenius notation* for λ.

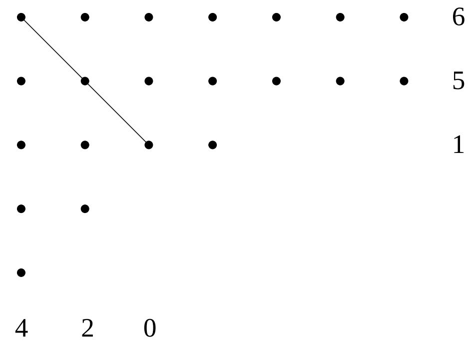


Figure 1.33: Frobenius notation

(b) Suppose that the path P consists of c_1 steps N, followed by c_2 steps E, then c_3 steps S, etc., ending in c_ℓ steps. If $\ell = 2r$ then associate with P the partition λ whose Frobenius notation is

$$A_\lambda = \begin{pmatrix} c_{\ell-1} & c_{\ell-3} & c_{\ell-5} & \cdots & c_1 \\ c_\ell - 1 & c_{\ell-2} - 1 & c_{\ell-4} - 1 & \cdots & c_2 - 1 \end{pmatrix}.$$

If $\ell = 2r - 1$ then associate with P the partition λ whose Frobenius notation is

$$A_\lambda = \begin{pmatrix} c_{\ell-1} & c_{\ell-3} & \cdots & c_2 & 0 \\ c_\ell - 1 & c_{\ell-2} - 1 & \cdots & c_3 - 1 & c_1 - 1 \end{pmatrix}.$$

This sets up the desired bijection. For instance, the CSSAW of Figure 1.34(a) corresponds to the partition $\lambda = (8, 6, 5, 2, 1)$ with $A_\lambda = \begin{pmatrix} 7 & 4 & 2 \\ 4 & 2 & 0 \end{pmatrix}$, while Figure 1.34(b) corresponds to $\lambda = (4, 3, 3, 3, 2, 1, 1)$ with $A_\lambda = \begin{pmatrix} 3 & 1 & 0 \\ 6 & 3 & 1 \end{pmatrix}$. This result is due to A. J. Guttman and M. D. Hirschhorn, *J. Phys. A Math. Gen.* **17** (1984), 3613–3614. They give a combinatorial proof equivalent to the above, though not stated in terms of Frobenius notation. The connection with Frobenius notation was given by G. E. Andrews, *Electronic J. Combinatorics* **18(2)** (2011), P6.

71. *Answer.* $p(0) + p(1) + \cdots + p(n)$. Given $\nu \vdash k \leq n$, define λ to be ν with the part $n - k$ adjoined (in the correct position, so the parts remain weakly decreasing), and define μ to be ν with $n - k + 1$ adjoined. This yields the desired bijection. For some generalizations, see Theorem 3.21.11 and Exercise 3.150.

72. This exercise gives a glimpse of the fascinating subject of *plane partitions*, treated extensively in Sections 7.20–7.22.

(a) Although equation (1.128) can be proved by *ad hoc* arguments, the "best" proof is a bijection using the RSK algorithm, the special case $q = 1$, $r = 2$ and $c \to \infty$ of Theorem 7.20.1. A different generalization, but with a non-bijective proof, is given by Theorem 7.21.7.

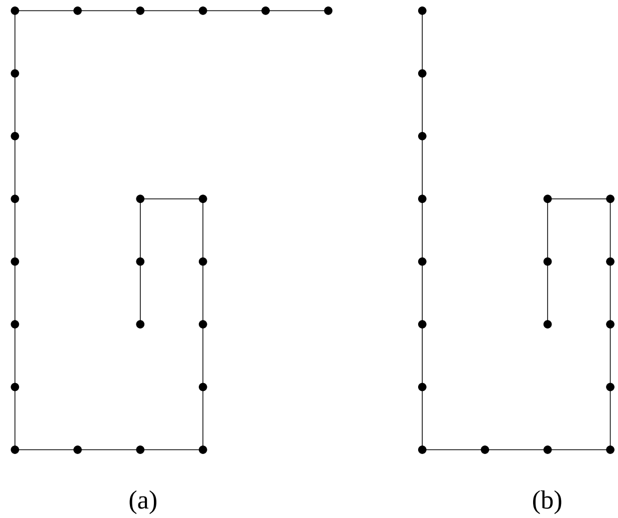
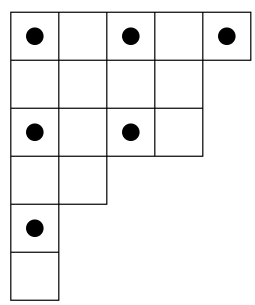


Figure 1.34: Two concatenated spiral self-avoiding walks

(b) This result is due to B. Gordon, *Proc. Amer. Math. Soc.* **13** (1962), 869–873. A bijective proof was given by C. Sudler, Jr., *Proc. Amer. Math. Soc.* **16** (1965), 161–168. This result can be generalized to a chain $\lambda^1 \subseteq \lambda^2 \subseteq \cdots \subseteq \lambda^k$ of any fixed number k of strict partitions, and with a fixed bound on the largest part of λ^k. See [7.146, Prop. 16.1] and G. E. Andrews, *Pacific J. Math.* **72** (1977), 283–291.

73. Consider for instance $\lambda = (5, 4, 4, 2, 1, 1)$, and put dots in the squares of the diagram of λ as follows:



Count the total number of dots by rows and by columns to obtain the first identity. The other formulas are analogous. There are many further variations.

74. Subtract one from each part of a partition of n into $n-t$ parts to deduce that $p_{n-t}(n) = p(t)$ if and only if $n \geq 2t$.

75. The partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ corresponds to $\lambda_1 + k - 1 > \lambda_2 + k - 2 > \cdots > \lambda_k$.

76. By the bijection illustrated in Figure 1.16, the coefficient of $q^k x^n$ in the left-hand side of equation (1.129) is equal to the number of self-conjugate partitions λ of n whose rank is k. If we remove the Durfee square from the diagram of λ, then we obtain two partitions μ and μ' (the conjugate of μ) with largest part at most k. Hence we obtain the right-hand side of (1.129).

One can also prove this identity by making the substitution $x \to x^2$ and $q \to qx^{-1}$ into equation (1.83).

77. Given $r \in \mathbb{Z}$, let λ be a partition satisfying $\lambda_1' + r \geq \lambda_1 - 1$. Define $\psi_r(\lambda)$ to be the partition obtained by removing the first column of (the diagram of) λ and adding a new row at the top of length $\lambda_1' + r$. We need to give a bijection

$$\gamma_n : \bigcup_{m \in 2\mathbb{Z}} \mathrm{Par}(n - m(3m-1)/2) \to \bigcup_{m \in 1+2\mathbb{Z}} \mathrm{Par}(n - m(3m-1)/2).$$

One can check that we can define γ_n as follows: for $\lambda \in \bigcup_{m \in 2\mathbb{Z}} \mathrm{Par}(n - m(3m-1)/2)$, let

$$\gamma_n(\lambda) = \begin{cases} \psi_{-3m-1}(\lambda), & \text{if } \lambda_1 - \lambda_1' + 3m \leq 0 \\ \psi^{-1}_{-3m+2}(\lambda), & \text{if } \lambda_1 - \lambda_1' + 3m \geq 0. \end{cases}$$

This proof appears in D. M. Bressoud and D. Zeilberger, *Amer. Math. Monthly* **92** (1985), 54–55. Our presentation follows Pak [1.62, §5.4.1].

78. (a) Some related results are due to Euler and recounted in [1.55, §303].

    (b) This problem was suggested by Dale Worley. For each $1 \leq i \leq n$, each partition λ of $n - i$, and each divisor d of i, we wish to associate a d-element multiset M of partitions of n so that every partition of n occurs exactly n times. Given i, λ, and d, simply associate d copies of the partition obtained by adjoining i/d d's to λ.

79. (a) See [1.2, Cor. 8.6].

    (b) Clearly $p_S(n) = 1$ for all n, so the statement $q_S(n) = 1$ is just the uniqueness of the binary expansion of n.

80. For each partition λ of n and each part j of λ occurring at least k times, we need to associate a partition μ of n such that the total number of times a given μ occurs is the same as the number $f_k(\mu)$ of parts of μ that are equal to k. To do this, simply change k of the j's in λ to j k's. For example, $n = 6$, $k = 2$:

λ	j	μ
1 1 1 1 1 1	1	2 1 1 1 1
2 1 1 1 1	1	2 2 1 1
3 1 1 1	1	3 2 1
4 1 1	1	4 2
2 2 1 1	2	2 2 1 1
2 2 1 1	1	2 2 2
2 2 2	2	2 2 2
3 3	3	2 2 2.

This result was discovered by R. Stanley in 1972 and submitted to the Problems and Solutions section of *Amer. Math. Monthly*. It was rejected with the comment "A bit on the easy side, and using only a standard argument." Daniel I. A. Cohen learned of this result and included the case $k = 1$ as Problem 75 of Chapter 3 in his book *Basic Techniques of Combinatorial Theory*, Wiley, New York, 1978. For this reason the case $k = 1$ is sometimes called "Stanley's theorem." The generalization from $k = 1$ to arbitrary k was independently found by Paul Elder in 1984, as reported by R. Honsberger, *Mathematical Gems III*, Mathematical Association of America, 1985 (page 8). For this reason the general case is sometimes called "Elder's theorem." An independent proof of the general case was given by M. S. Kirdar and T. H. R. Skyrme, *Canad. J. Math.* **34** (1982), 194–195, based on generating functions. The bijection given here also appears in A. H. M. Hoare, *Amer. Math. Monthly* **93** (1986), 475–476. Another proof appears in L. Solomon, *Istituto Nazionale di Alta Matematica, Symposia Matematica*, vol. 13 (1974), 453–466 (lemma on p. 461).

81. Given an ordered factorization $n+1 = a_1 a_2 \cdots a_k$, set $a_0 = 1$ and let λ be the partition for which the part $a_0 a_1 \cdots a_{j-1}$ occurs with multiplicity $a_j - 1$, $1 \leq j \leq k$. For instance, if $24 = 3 \cdot 2 \cdot 4$ then we obtain the partition 666311 of 23. This procedure sets up a bijection with perfect partitions of n, due to P. A. MacMahon, *Messenger Math.* **20** (1891), 103–119; reprinted in [1.3, pp. 771–787]. Note that if we have a perfect partition λ of n with largest part m, then there are exactly two ways to add a part p to λ to obtain another perfect partition, viz., $p = m$ or $p = n + 1$.

82. This result is due to S. Ramanujan in 1919, who obtained the remarkable identity

$$\sum_{n\geq 0} p(5n+4)x^n = 5\frac{\prod_{k\geq 1}(1-x^{5k})^5}{\prod_{k\geq 1}(1-x^k)^6}.$$

F. J. Dyson conjectured in 1944 that for each $0 \leq i \leq 4$, exactly $p(5n+4)/5$ partitions λ of $5n + 4$ satisfy $\lambda_1 - \lambda_1' \equiv i \pmod 5$. This conjecture was proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer in 1953. Many generalizations of these results are known. For an introduction to the subject of partition congruences, see Andrews [1.2, Ch. 10]. For more recent work in this area, see K. Mahlburg, *Proc. National Acad. Sci.* **102** (2005), 15373–15376.

83. *Some hints.* Let A be the set of all partitions λ such that $\lambda_{2i-1} - \lambda_{2i} \leq 1$ for all i, and let B be the set of all partitions λ such that λ' has only odd parts, each of which is repeated an even number of times. Verify the following statements.

- There is bijection $A \times B \to \mathrm{Par}$ satisfying $w(\mu)w(\nu) = w(\lambda)$ if $(\mu, \nu) \mapsto \lambda$.
- We have
$$\sum_{\lambda\in B} w(\lambda) = \prod_{j\geq 1} \frac{1}{1 - a^j b^j c^{j-1} d^{j-1}}.$$
- Let $\lambda \in A$. Then the pairs $(\lambda_{2i-1}, \lambda_{2i})$ fall into two classes: (a, a) (which can occur any number of times), and $(a+1, a)$ (which can occur at most once). Deduce that
$$\sum_{\lambda\in A} w(\lambda) = \prod_{j\geq 1} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^j c^{j-1} d^{j-1})}.$$

This elegant bijective proof is due to C. Boulet, *Ramanujan J.* **3** (2006), 315–320, simplifying and generalizing previous work of G. E. Andrews, A. V. Sills, R. P. Stanley, and A. J. Yee.

88. (a) These are the famous *Rogers-Ramanujan identities*, first proved by L. J. Rogers, *Proc. London Math. Soc.* **25** (1894), 318–343, and later rediscovered by I. Schur, *Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Klasse* (1917), 302–321, S. Ramanujan (sometime before 1913, without proof), and others. For a non-combinatorial proof, see e.g. [1.2, §7.3]. For an exposition and discussion of bijective proofs, see Pak [1.62, §7 and pp. 62–63]. For an interesting recent bijective proof, see C. Boulet and I. Pak, *J. Combinatorial Theory, Ser. A* **113** (2006), 1019–1030. None of the known bijective proofs of the Rogers-Ramanujan identities can be considered "simple," comparable to the proof we have given of the pentagonal number formula (Proposition 1.8.7). An interesting reason for the impossibility of a nice proof was given by I. Pak, The nature of partition bijections II. Asymptotic stability, preprint.

    (b) These combinatorial interpretations of the Rogers-Ramanujan identities are due to P. A. MacMahon, [1.55, §§276–280]. They can be proved similarly to the proof of Proposition 1.8.6, based on the observation that $(\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition of n with at most k parts if and only if $(\lambda_1 + 2k - 1, \lambda_2 + 2k - 3, \dots, \lambda_k + 1)$ is a partition of $n + k^2$ whose parts differ by at least two and with exactly k parts, and similarly for $(\lambda_1 + 2k, \lambda_2 + 2k - 2, \dots, \lambda_k + 2)$.

86. This is *Schur's partition theorem.* See G. E. Andrews, in *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, American Mathematical Society, Providence, RI, 1986, pp. 53–58. For a bijective proof, see D. M. Bressoud, *Proc. Amer. Math. Soc.* **79** (1980), 338–340. It is surprising that Schur's partition theorem is easier to prove bijectively than the Rogers-Ramanujan identities (Exercise 1.88).

89. Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of n into k odd parts less than $2k$. We begin with the lecture hall partition $\lambda^0 = (0, \dots, 0)$ of length k and successively insert the parts $\mu_1, \mu_2, \dots, \mu_k$ to build up a sequence of lecture hall partitions $\lambda^1, \lambda^2, \dots, \lambda^k = \lambda$. The rule for inserting $\mu_i := 2\nu_i - 1$ into λ^{i-1} is the following. Add 1 to the parts of λ^{i-1} (allowing 0 as a part), beginning with the largest, until either (i) we have added 1 to μ_i parts of λ_{i-1}, or (ii) we encounter a value λ^{i-1}_{2c-1} for which

$$\frac{\lambda^{i-1}_{2c-1}}{n - 2c + 2} = \frac{\lambda^{i-1}_{2c}}{n - 2c + 1}.$$

In this case we add $\nu_i - c + 1$ to λ^{i-1}_{2c-1} and $\nu_i - c$ to λ^{i-1}_{2c}. It can then be checked that the map $\mu \mapsto \lambda$ gives the desired bijection.

Example. Let $k = 5$ and $\mu = (7, 5, 5, 3, 1)$. We have $\frac{\lambda^0_1}{5} = \frac{\lambda^0_2}{4} = 0$. Hence $\lambda^1 = (4, 3, 0, 0, 0)$. We now have $\frac{\lambda^1_1}{5} \neq \frac{\lambda^1_2}{4}$, but $\frac{\lambda^1_3}{3} = \frac{\lambda^1_4}{2} = 0$. Hence $\lambda^2 = (5, 4, 2, 1, 0)$. Continuing in this way we get $\lambda^3 = (8, 6, 2, 1, 0)$, $\lambda^4 = (9, 7, 3, 1, 0)$, and $\lambda = \lambda^5 = (10, 7, 3, 1, 0)$.

Lecture hall partitions were introduced by M. Bousquet-Mélou and K. Eriksson, *Ramanujan J.* **1** (1997), 101–111, 165–185. They proved the result of this exercise as well as many generalizations and refinements. In our sketch above we have followed A. J. Yee, *Ramanujan J.* **5** (2001), 247–262. Her bijection is a simplified description of the bijection of Bousquet-Mélou and Eriksson. Much further work has been done in this area; see e.g. S. Corteel and C. D. Savage, *J. Combinatorial Theory, Ser. A* **108** (2004), 217–245, for further information and references.

90. This curious result is connected with the theory of lecture hall partitions (Exercise 1.89). It was originally proved by M. Bousquet-Mélou and K. Eriksson, *Ramanujan J.* **1** (1997), 165–185 (end of Section 4). For a nice bijective proof of this result and related results, see C. D. Savage and A. J. Yee, *J. Combinatorial Theory, Ser. A* **115** (2008), 967–996.

91. (a) This famous result is the *Jacobi triple product identity*. It was first stated by C. F. Gauss (unpublished). The first published proof is due to C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Regiomonti, fratrum Bornträger, 1829; reprinted in *Gesammelte Werke*, vol. 1, Reimer, Berlin, 1881, pp. 49–239. For a summary of its bijective proofs, see Pak [1.62, §6 and pp. 60–62].

(b) Substitute $q^{3/2}$ for q and $-q^{1/2}$ for x, and simplify.

(c) For the first, set $x = -1$ and use equation (1.81). For the second, substitute $q^{1/2}$ for both x and q. The right-hand side then has a factor equal to 2. Divide both sides by 2 and again use equation (1.81). These identities are due to Gauss, Zur Theorie der neuen Transscendenten II, *Werke*, Band III, Göttingen, 1866, pp. 436–445 (§4). For a cancellation proof, see Exercise 2.31. For another proof of equation (1.132) based on counting partitions of n with empty 2-core, see Exercise 7.59(g).

(d) After making the suggested substitution we obtain

$$\sum_{n\in\mathbb{Z}}(-1)^n x^n q^{\binom{n}{2}} = \prod_{k\geq 1}(1-q^k)(1-xq^{k-1})(1-x^{-1}q^k).$$

Rewrite the left-hand side as

$$1+\sum_{n\geq 1}(-1)^n(x^{-n}+x^n)q^{\binom{n}{2}}.$$

Now divide both sides by $1-x$ and let $x\to 1$. The left-hand side becomes $\sum_{n\geq 0}(-1)^n(2n+1)q^{\binom{n}{2}}$. The right-hand side has a factor equal to $1-x$, so deleting this factor and then setting $x=1$ gives

$$(1-q)^2\prod_{k\geq 2}(1-q^{k-1})(1-q^k)^2 = \prod_{k\geq 1}(1-q^k)^3,$$

and the proof follows. This identity is due to C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Regiomonti, Sumtibus fratrum Borntraeger, Königsberg, Germany, 1829 (page 90).

92. This identity is due to G. E. Andrews, *Amer. Math. Monthly* **94** (1987), 437–439. A simple proof based on the Jacobi triple product identity (Exercise 1.91) is due to F. G. Garvan, in *Number Theory for the Millenium, II* (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 75–92 (§1). This paper contains many further similar identities. For a continuation, see F. G. Garvan and H. Yesilyurt, *Int. J. Number Theory* **3** (2007), 1–42. No bijective proofs are known of any of these identities.

93. This identity is due to F. G. Garvan, *op. cit.* This paper and the continuation by Garvan and Yesilyurt, *op. cit.*, contain many similar identities. No bijective proofs are known of any of them.

94. The sequence $a_1, a_2, \ldots$ (sometimes prepended with $a_0 = 0$) is called *Stern's diatomic sequence*, after the paper M. A. Stern, *J. Reine angew. Math.* **55** (1858), 193–220. For a survey of its remarkable properties, see S. Northshield, *Amer. Math. Monthly* **117** (2010), 581–598.

95. This remarkable result is due to D. Applegate, O. E. Pol, and N. J. A. Sloane, *Congressus Numerantium* **206** (2010), 157–191.

96. (a) The function $\tau(n)$ is *Ramanujan's tau function.* The function

$$\Delta(t) = (2\pi)^{12} \sum_{n\geq 1} \tau(n) e^{2\pi i t}$$

plays an important role in the theory of modular forms; see e.g. T. Apostol, *Modular Forms and Dirichlet Series in Number Theory* 2nd, ed., Springer-Verlag, New York, 1997 (p. 20) or J.-P. Serre, *A Course in Arithmetic*, Springer-Verlag, New York, 1973 (§VII.4). The multiplicativity property of this exercise was conjectured by S. Ramanujan, *Trans. Cambridge Phil. Soc.* **22** (1916), 159–184, and proved by L. J. Mordell, *Proc. Cambridge Phil. Soc.* **19** (1917), 117–124.

(b) This result was also conjectured by Ramanujan, *op. cit.*, and proved by Mordell, *op. cit.*

(c) This inequality was conjectured by Ramanujan, *op. cit.*, and proved by P. R. Deligne, *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307; **52** (1980), 137–252. Deligne deduced Ramanujan's conjecture (in a nontrivial way) from his proof of the Riemann hypothesis for varieties over finite fields (the most difficult part of the "Weil conjectures"). Deligne in fact proved a conjecture of Petersson generalizing Ramanujan's conjecture.

(d) This inequality was conjectured by D. H. Lehmer, *Duke Math. J.* **14** (1947), 429–492. It is known to be true for (at least) $n < 2.2 \times 10^{16}$.

97. This result follows from the case $p = 2$ and $\mu = \emptyset$ of Exercise 7.59(e). Greta Panova (October 2007) observed that it can also be deduced from Exercise 1.83. Namely, first prove by induction that the Ferrers diagram of λ can be covered by edges if and only if the Young diagram of λ has the same number of white squares as black squares in the usual chessboard coloring. Thus $f(n)$ is the coefficient of q^n in the right-hand side

of equation (1.130) after substituting $a = d = q/y$ and $b = c = y$. Apply the Jacobi triple product identity (Exercise 1.91) to the numerator and then set $y = 0$ to get $\sum_{n\geq 0} f(n)q^n = 1/\prod_{j\geq 1}(1-q^j)^2$.

98. Substitute na for j, $-x$ for x, and ζ for q in the q-binomial theorem (equation (1.87)). The proof follows straightforwardly from the identity

$$\prod_{m=0}^{na-1}(1-\zeta^m x) = (1-x^n)^a.$$

For a host of generalizations, see V. Reiner, D. Stanton, and D. White, *J. Combinatorial Theory, Ser. A* **108** (2004), 17–50.

99. It is an immediate consequence of the identity $f(q) = q^{k(n-k)}f(1/q)$ that

$$f'(1) = \frac{1}{2}k(n-k)f(1) = \frac{1}{2}k(n-k)\binom{n}{k}.$$

100. The Chu-Vandermonde identity follows from $(1+x)^{a+b} = (1+x)^a(1+x)^b$. Write $f_n(x) = (1+x)(1+qx)\cdots(1+q^{n-1}x)$. The q-analogue of $(1+x)^{a+b} = (1+x)^a(1+x)^b$ is $f_{a+b}(x) = f_b(x)f_a(q^b x)$. By the q-binomial theorem (equation (1.87)) we get

$$\sum_{n=0}^{a+b} q^{\binom{n}{2}}\binom{\boldsymbol{a}+\boldsymbol{b}}{\boldsymbol{n}}x^n = \left(\sum_{k=0}^{b} q^{\binom{k}{2}}\binom{\boldsymbol{b}}{\boldsymbol{k}}x^k\right)\left(\sum_{k=0}^{a} q^{bk+\binom{k}{2}}\binom{\boldsymbol{a}}{\boldsymbol{k}}x^k\right).$$

Equating coefficients of x^n yields

$$\begin{aligned} q^{\binom{n}{2}}\binom{\boldsymbol{a}+\boldsymbol{b}}{\boldsymbol{n}} &= \sum_{k=0}^{n} q^{\binom{n-k}{2}+bk+\binom{k}{2}}\binom{\boldsymbol{b}}{\boldsymbol{n}-\boldsymbol{k}}\binom{\boldsymbol{a}}{\boldsymbol{k}} \\ \Rightarrow \binom{\boldsymbol{a}+\boldsymbol{b}}{\boldsymbol{n}} &= \sum_{k=0}^{n} q^{k(k+b-n)}\binom{\boldsymbol{a}}{\boldsymbol{k}}\binom{\boldsymbol{b}}{\boldsymbol{n}-\boldsymbol{k}}. \end{aligned}$$

103. See Lemma 3.1 of K. Liu, C. H. F. Yan, and J. Zhou, *Sci. China, Ser. A* **45** (2002), 420–431, for a proof based on the Hilbert scheme of n points in the plane. A combinatorial proof of a continuous family of results including this exercise appears in N. Loehr and G. S. Warrington, *J. Combinatorial Theory, Ser. A* **116** (2009), 379–403.

104. Let $f(x) = 1 + x + \cdots + x^9$ and $i^2 = 1$. It is not hard to see that

$$\begin{aligned} f(n) &= \frac{1}{4}\left(f(1)^n + f(i)^n + f(-1)^n + f(-i)^n\right) \\ &= \frac{1}{4}(10^n + (1+i)^n + (1-i)^n) \\ &= \begin{cases} \frac{1}{4}(10^n + (-1)^k 2^{2k-1}), & n = 4k \\ \frac{1}{4}(10^n + (-1)^k 2^{2k-1}), & n = 4k+1 \\ \frac{1}{4}10^n, & n = 4k+2 \\ \frac{1}{4}(10^n + (-1)^{k+1}2^{2k}), & n = 4k+3. \end{cases} \end{aligned}$$

105. (a) Let $P(x) = (1+x)(1+x^2)\cdots(1+x^n) = \sum_{k\geq 0} a_k x^k$. Let $\zeta = e^{2\pi i/n}$ (or any primitive nth root of unity). Since for any integer k,

$$\sum_{j=1}^{n} \zeta^{kj} = \begin{cases} n, & \text{if } n|k \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\frac{1}{n}\sum_{j=1}^{n} P(\zeta^j) = \sum_j a_{jn} = f(n).$$

Now if ζ^j is a primitive dth root of unity (so $d = n/(j,n)$), then

$$x^d - 1 = (x-\zeta^j)(x-\zeta^{2j})\cdots(x-\zeta^{dj}),$$

so putting $x = -1$ yields

$$(1+\zeta^j)(1+\zeta^{2j})\cdots(1+\zeta^{dj}) = \begin{cases} 2, & d \text{ odd} \\ 0, & d \text{ even.} \end{cases}$$

Hence

$$P(\zeta^j) = \begin{cases} 2^{n/d}, & d \text{ odd} \\ 0, & d \text{ even.} \end{cases}$$

Since there are $\phi(d)$ values of $j \in [n]$ for which ζ^j is a primitive dth root of unity, we obtain

$$f(n) = \frac{1}{n}\sum_{j=1}^{n} P(\zeta^j) = \frac{1}{n}\sum_{\substack{d|n \\ d \text{ odd}}} \phi(d) 2^{n/d}.$$

This result appears in R. Stanley and M. F. Yoder, *JPL Technical Report 32-1526*, Deep Space Network **14** (1972), 117–123.

(b) Suppose that n is an odd prime. Identify the beads of a necklace with $\mathbb{Z}/n\mathbb{Z}$ in an obvious way. Let $S \subseteq \mathbb{Z}/n\mathbb{Z}$ be the set of black beads. If $S \neq \emptyset$ and $S \neq \mathbb{Z}/n\mathbb{Z}$, then there is a unique $a \in \mathbb{Z}/n\mathbb{Z}$ for which

$$\sum_{x\in S}(x+a) = 0.$$

The set $\{x+a : x \in S\}$ represents the same necklace (up to cyclic symmetry), so we have associated with each non-monochromatic necklace a subset of $\mathbb{Z}/n\mathbb{Z}$ whose elements sum to 0. Associate with the necklaces of all black beads and all white beads the subsets $S = \emptyset$ and and $S = \mathbb{Z}/n\mathbb{Z}$, and we have the desired bijection.

A proof for any odd n avoiding roots of unity and generating functions was given by Anders Kaseorg (private communication) in 2004, though the proof is not a direct bijection.

(c) See A. M. Odlyzko and R. Stanley, *J. Number Theory* **10** (1978), 263–272.

106. We claim that $f(n,k)$ is just the Stirling number $S(n,k)$ of the second kind. We need to associate with a sequence $a_1 \cdots a_n$ being counted a partition of $[n]$ into k blocks. Simply put i and j in the same block when $a_i = a_j$. This yields the desired bijection. The sequences $a_1 \cdots a_n$ are called *restricted growth functions* or *restricted growth strings* (sometimes with 1 subtracted from each term). For further information, see S. Milne, *Advances in Math.* **26** (1977), 290–305.

108. (a) Given a partition π of $[n-1]$, let $i, i+1, \dots, j$ for $j > i$, be a maximal sequence of two or more consecutive integers contained in a block of π. Remove $j-1$, $j-3$, $j-5, \dots$ from this sequence and put them in a block with n. Doing this for every such sequence i, $i+1, \dots, j$ yields the desired bijection. See H. Prodinger, *Fibonacci Quart.* **19** (1981), 463–465, W. Y. C. Chen, E. Y. P. Deng, and R. R. X. Du, *Europ. J. Combin.* **26** (2005), 237–243, and W. Yang, *Discrete Math.* **156** (1996), 247–252.

    Example. If π = 1456-2378, then the bijection gives 146-38-2579.

    The above proof easily extends (as done in papers cited above) to show the following result: let $0 \le k \le n$, and let $B_k(n)$ be the number of partitions of $[n]$ so that if i and j are in a block then $|i-j| > k$. Then $B_k(n) = B(n-k)$.

109. (a) Given a partition $\pi \in \Pi_n$, list the blocks in decreasing order of their smallest element. Then list the elements of each block with the least element first, followed by the remaining elements in decreasing order, obtaining a permutation $w \in \mathfrak{S}_n$. The map $\pi \mapsto w$ is bijection from Π_n to the permutations being enumerated. For instance, if $\pi = 13569 - 248 - 7$, then $w = 728419653$. To obtain π from w, break w before each left-to-right minimum. This result, as well as those in (b) and (c), is due to A. Claesson, *Europ. J. Combinatorics* **22** (2001), 961–971.

    (b) Now write the blocks in decreasing order of their smallest element, with the elements of each block written in increasing order.

    (c) Let w be the permutation corresponding to π as defined in (a). Then w also satisfies the condition of (b) if and only if each block of π has size one or two.

110. *Answer:* the coefficient of x^n is $B(n-1)$, $n \ge 1$. See Proposition 2.6 of M. Klazar, *J. Combinatorial Theory, Ser. A* **102** (2003), 63–87.

111. The number of ways to partition a k-element subset of $[n]$ into j intervals is $\binom{k-1}{j-1}\binom{n-k+j}{j}$, since we can choose the interval sizes from left-to-right in $\binom{k-1}{j-1}$ ways (the number of compositions of k into j parts), and then choose the intervals themselves in $\binom{n-k+j}{j}$ ways. Hence by the Principle of Inclusion-Exclusion (Theorem 2.1.1),

$$f(n) = B(n) + \sum_{k=1}^{n} \sum_{j=1}^{k} B(n-k)(-1)^j \binom{k-1}{j-1}\binom{n-k+j}{j}.$$

Now

$$\sum_{j=1}^{k} (-1)^j \binom{k-1}{j-1}\binom{n-k+j}{j} = (-1)^k \binom{n-k+1}{k}.$$

Hence

$$\begin{aligned} f(n) &= \sum_{k=0}^{n} B(n-k)(-1)^k \binom{n-k+1}{k} \\ &= \sum_{k=0}^{n} B(k)(-1)^{n-k} \binom{k+1}{n-k}. \end{aligned}$$

(Is there some way to see this directly from Inclusion-Exclusion?) Now multiply by x^n and sum on $n \geq 0$. Since by the binomial theorem

$$\sum_{n\geq 0} (-1)^{n-k} \binom{k+1}{n-k} x^n = x^k (1-x)^{k+1},$$

we get

$$\begin{aligned} F(x) &= \sum_{k\geq 0} B(k) x^k (1-x)^{k+1} \\ &= (1-x) G(x(1-x)). \end{aligned}$$

115. See D. Chebikin, R. Ehrenborg, P. Pylyavskyy, and M. A. Readdy, *J. Combinatorial Theory, Ser. A* **116** (2009), 247–264. The polynomials $Q_n(t)$ are introduced in this paper and are shown to have many cyclotomic factors, but many additional such factors are not yet understood.

116. (b) See L. A. Shepp and S. P. Lloyd, *Trans. Amer. Math. Soc.* **121** (1966), 340–357.

117. *Answer.* $p_{nk} = 1/n$ for $1 \leq k \leq n$. To see this, consider the permutations $v = b_1 \cdots b_{n+1}$ of $[n] \cup \{*\}$ beginning with 1. Put the elements to the left of $*$ in a cycle in the order they occur. Regard the elements to the right of $*$ as a word which defines a permutation of its elements (say with respect to the elements listed in increasing order). This defines a bijection between the permutations v and the permutations $w \in \mathfrak{S}_n$. The length of the cycle containing 1 is k if $b_{k+1} = *$. Since $*$ is equally likely to be any of $b_2, \ldots, b_{n+1}$, the proof follows.

Example. Let $v = 1652{*}4873$. Then w has the cycle $(1,6,5,2)$. The remaining elements are permuted as 4873 with respect to the increasing order 3478, i.e., $w(3) = 4$, $w(4) = 8$, $w(7) = 7$, and $w(8) = 3$. In cycle form, we have $w = (1,6,5,2)(3,4,8)(7)$.

118. (b) We compute equivalently the probability that $n, n-1, \ldots, n-\lambda_1+1$ are in the same cycle C_1, and $n-\lambda_1, \ldots, n-\lambda_1-\lambda_2+1$ are in the same cycle C_2 different from C_1, etc. Apply the fundamental bijection of Proposition 1.3.1 to w, obtaining a permutation $v = b_1 \cdots b_n$. It is easy to check that w has the desired properties if and only if the restriction u of v to $n-k+1, n-k+2, \ldots, n$ has $n-k+\lambda_\ell$ appearing first, then the elements $n-k+1, n-k+2, \ldots, n-k+\lambda_\ell-1$ in some order, then $n-k+\lambda_{\ell-1}+\lambda_\ell$, then the elements $n-k+\lambda_\ell+1, \ldots, n-k+\lambda_{\ell-1}+\lambda_\ell+1$ in some order, then $n-k+\lambda_{\ell-2}+\lambda_{\ell-1}+\lambda_\ell$, etc. Hence of the $k!$ permutations of $n-k+1, \ldots, n$ there are $(\lambda_1-1)! \cdots (\lambda_\ell-1)!$ choices for u, and the proof follows. For a variant of this problem when the distribution isn't uniform, see R. X. Du and R. Stanley, in preparation.

(c) Let v be as in (b), and let $v' = b_2b_1b_3b_4\cdots b_n$. Exactly one of v and v' is even. Moreover, the condition in (b) on the restriction u is unaffected unless $b_1 = n-k+\lambda_\ell$ and $b_2 = n-k+i$ for some $1 \leq i \leq \lambda_\ell - 1$. In this case v has exactly ℓ records, so w has exactly ℓ cycles. Hence w is even if and only if $n-\ell$ is even. Moreover, the number of choices for u is

$$\frac{(n-2)!}{(k-2)!}(\lambda_1-1)!\cdots(\lambda_\ell-1)!,$$

and the proof follows easily.

119. If a permutation $w \in \mathfrak{S}_{2n}$ has a cycle C of length $k > n$, then it has exactly one such cycle. There are $\binom{2n}{k}$ ways to choose the elements of C, then $(k-1)!$ ways to choose C, and finally $(2n-k)!$ ways to choose the remainder of w. Hence

$$\begin{aligned} P_n &= 1 - \frac{1}{(2n)!}\sum_{k=n+1}^{2n}\binom{2n}{k}(k-1)!(2n-k)! \\ &= 1 - \sum_{k=n+1}^{2n}\frac{1}{k} \\ &= 1 - \sum_{k=1}^{2n}\frac{1}{k} + \sum_{k=1}^{n}\frac{1}{k} \\ &\sim 1 - \log(2n) + \log(n) \\ &= 1 - \log 2, \end{aligned}$$

and the proof follows. For an amusing application of this result, see P. M. Winkler, *Mathematical Mind-Benders*, A K Peters, Wellesley, MA, 2007 (pp. 12, 18–20).

120. *First solution.* There are $\binom{n}{k}(k-1)!$ k-cycles, and each occurs in $(n-k)!$ permutations $w \in \mathfrak{S}_n$. Hence

$$E_k(n) = \frac{1}{n!}\binom{n}{k}(k-1)!(n-k)! = \frac{1}{k}.$$

Second solution. By Exercise 1.117 (for which we gave a simple bijective proof) the probability that some element $i \in [n]$ is in a k-cycle is $1/n$. Since there are n elements and each k-cycle contains k of them, the expected number of k-cycles is $(1/n)(n/k) = 1/k$.

124. (a) Let $w = a_1a_2\cdots a_{n+1} \in \mathfrak{S}_{n+1}$ have k inversions, where $n \geq k$. There are $f_k(n)$ such w with $a_{n+1} = n+1$. If $a_i = n+1$ with $i < n+1$, then we can interchange a_i and a_{i+1} to form a permutation $w' \in \mathfrak{S}_{n+1}$ with $k-1$ inversions. Since $n \geq k$, every $w' = b_1b_2\cdots b_{n+1} \in \mathfrak{S}_{n+1}$ with $k-1$ inversions satisfies $b_1 \neq n+1$ and thus can be obtained from a $w \in \mathfrak{S}_{n+1}$ with k inversions as above.

(b) Use induction on k.

(c) By Corollary 1.3.13 we have

$$\begin{aligned}\sum_{k\geq 0} f_k(n)q^k &= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})\\ &= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}\\ &= (1-q)(1-q^2)\cdots(1-q^n)\sum_{k\geq 0}\binom{-n}{k}(-1)^k q^k.\end{aligned}$$

Hence if $\prod_{i\geq 1}(1-q^i)=\sum_{j\geq 0} b_j q^j$, then

$$f_k(n)=\sum_{j=0}^{k}(-1)^j b_{k-j},\quad n\geq k.$$

Moreover, it follows from the Pentagonal Number Formula (1.88) that

$$b_r=\begin{cases}(-1)^i, & \text{if } r=i(3i\pm 1)/2\\ 0, & \text{otherwise.}\end{cases}$$

See pp. 15–16 of D. E. Knuth [1.48].

127. (a) We can reason analogously to the proofs of Proposition 1.3.12 and Corollary 1.3.13. Given $w=w_1w_2\cdots w_n\in\mathfrak{S}_n$ and $1\leq i\leq n$, define

$$r_i=\#\{j\,:\,j<i,\ w_j>w_i\}$$

and $\mathrm{code}'(w)=(r_1,\ldots,r_n)$. For instance, $\mathrm{code}'(3265174)=(0,1,0,1,4,0,3)$. Note that $\mathrm{code}'(w)$ is just a variant of $\mathrm{code}(w)$ and gives a bijection from $\mathfrak{S}_n$ to sequences $(r_1,\ldots,r_n)$ satisfying $0\leq r_i\leq i-1$. Moreover, $\mathrm{inv}(w)=\sum r_i$, and w_i is a left-to-right maximum if and only if $r_i=0$. From these observations equation (1.135) is immediate.

(b) Let $I(w)=(a_1,\ldots,a_n)$, the inversion table of w. Then $\mathrm{inv}(w)=\sum a_i$ (as noted in the proof of Corollary 1.3.13), and i is the value of a record if and only if $a_i=0$. From these observations equation (1.136) is immediate.

128. (a) First establish the recurrence

$$\sum_{j=1}^{n} f(j)(n-j)!=n!,\quad n\geq 1,$$

where we set $g(0)=1$. Then multiply by x^n and sum on $n\geq 0$. This result appears in L. Comtet, *Comptes Rend. Acad. Sci. Paris* **A 275** (1972), 569–572, and is also considered by Comtet in his book *Advanced Combinatorics*, Reidel, Dordrecht/Boston, 1974 (Exercise VII.16). For an extension of this exercise and further references, see Exercise 2.13.

(b) (I. M. Gessel) Now we have

$$n! = g(n) + \sum_{j=1}^{n} g(j-1)(n-j)!, \quad n \geq 1,$$

where we set $g(0) = 1$.

(c) See D. Callan, *J. Integer Sequences* **7** (2004), article 04.1.8.

(d) See M. H. Albert, M. D. Atkinson, and M. Klazar, *J. Integer Sequences* **6** (2003), article 02.4.4. For a survey of simple permutations, see R. Brignall, in *Permutation Patterns (2010)* (S. Linton, N. Ruškuc and V. Vatter, eds.), London Mathematical Society Lecture Note Series, vol. 376, Cambridge University Press, pp. 41–65. For some analogous results for set partitions, see M. Klazar, *J. Combinatorial Theory, Ser. A* **102** 2003), 63–87.

129. (b) It is easy to see that if w is an indecomposable permutation in $\mathfrak{S}_n$ with k inversions, then $n \leq k+1$. (Moreover, there are exactly 2^{k-1} indecomposable permutations in $\mathfrak{S}_{k+1}$ with k inversions.) Hence $g_n(q)$ has smallest term of degree $n-1$, and the proof follows.

(c) *Answer:* we have the continued fraction

$$1 - \frac{1}{F(q,x)} = \cfrac{a_0}{1 - \cfrac{a_1}{1 - \cfrac{a_2}{1 - \cdots}}},$$

where

$$a_n = (q^{\lfloor (n+1)/2 \rfloor} + q^{\lfloor (n+1)/2 \rfloor + 1} + \cdots + q^n)x.$$

See A. de Medicis and X. G. Viennot, *Advances in Appl. Math.* **15** (1994), 262–304 (equations (1.24) and (1.25), and Theorem 5.3).

130. This result, stated in a less elegant form, is due to M. Abramson and W. O. J. Moser, *Ann. Math. Statist.* **38** (1967), 1245–1254. The solution in the form of equation (1.138) is due to L. W. Shapiro and A. B. Stephens, *SIAM J. Discrete Math.* **4** (1991), 275–280.

133. (a) We have $\frac{1}{2}A_n(2) = \sum_{k=0}^{n-1} A(n,k+1)2^k$, where $A(n,k+1)$ permutations of $[n]$ have k descents. Thus we need to associate an ordered partition τ of $[n]$ with a pair (w,S), where $w \in \mathfrak{S}_n$ and $S \subseteq D(w)$. Given $w = a_1a_2\cdots a_n$, draw a vertical bar between a_i and a_{i+1} if $a_i < a_{i+1}$ or if $a_i > a_{i+1}$ and $i \in S$. The sets contained between bars (including the beginning and end) are read from left to right and define τ.

Example. Let $w = 724531968$ and $S = \{1,5\}$. Write $7|2|4|53|1|96|8$, so $\tau = (7,2,4,35,1,69,8)$.

134. See D. Foata and M.-P. Schützenberger, [1.26, Thm. 5.6]. For a vast generalization of this kind of formula, see E. Nevo and T. K. Petersen. *Discrete Computational Geometry* **45** (2011), 503–521.

135. (a) Put $x = -1$ in equation (1.40) and compare with (1.54).

(b) Let $n = 2m+1$. Since $\mathrm{des}(w) = m$ if w is alternating, it suffices to show combinatorially that $\sum_w (-1)^{\mathrm{des}(w)} = 0$, where w ranges over all non-alternating permutations in $\mathfrak{S}_n$. For a non-alternating permutation $w \in \mathfrak{S}_n$ let $T = T(w)$ be the increasing binary tree corresponding to w, as defined in Section 1.5. Since w is not alternating, it follows from the table preceding Proposition 1.5.3 that T has a vertex j with only one successor. For definiteness choose the least such vertex j, and let T' be the flip of T at j, as defined in Subsection 1.6.2. Define $w' \in \mathfrak{S}_n$ by $T(w') = T'$. Clearly $w'' = w$, so we have defined an involution $w \mapsto w'$ on all non-alternating permutations in $\mathfrak{S}_n$. Since n is odd, it again follows from the table preceding Proposition 1.5.3 that $\mathrm{des}(w)$ is the number of vertices of $T(w)$ with a left successor. Hence $(-1)^{\mathrm{des}(w)} + (-1)^{\mathrm{des}(w')} = 0$, and the proof follows. For further aspects of this line of reasoning, see D. Foata and M.-P. Schützenberger, [1.26, Thm. 5.6].

136. *Answer:* $c_1 = c_{n-1} = 1$, all other $c_i = 0$.

137. The number of $w \in \mathfrak{S}_n$ of type $\boldsymbol{c}$ is $\tau(\boldsymbol{c}) = n!/1^{c_1}c_1! \cdots n^{c_n}c_n!$. Let $n = a_0 + a_1\ell$. It is not hard to see that $\tau(\boldsymbol{c})$ is prime to ℓ if and only if, setting $k = c_\ell$, we have $c_1 \geq (n_1 - k)\ell$ where $\binom{n_1}{k}$ is prime to ℓ It follows from Exercise 1.14 that the number of binomial coefficients $\binom{n_1}{k}$ prime to ℓ is $\prod_{i\geq 1}(a_i + 1)$. Since $(c_1 - (n_1 - k)\ell, c_2, \ldots, c_{\ell-1})$ can be the type of an arbitrary partition of a_0, the proof follows.

This result first appeared in I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1979; second ed., 1995 (Ex. 10 of Ch. I.2). The proof given here appears on pp. 260–261 of R. Stanley, *Bull. Amer. Math. Soc.* **4** (1981), 254–265.

139. Let $z = \sum_{n\geq 1} g(n)x^n/n!$. Then $z' = 1 + \frac{1}{2}z^2 + \frac{1}{4!}z^4 + \cdots = \cosh(z)$. The solution to this differential equation satisfying $z(0) = 0$ is

$$z(x) = \log(\sec x + \tan x).$$

Since $z'(x) = \sec x$, it follows easily that $g(2n+1) = E_{2n}$. For further information and a bijective proof, see Section 3 of A. G. Kuznetsov, I. M. Pak, and A. E. Postnikov, [1.51].

140. *Hint.* Let $f_k(n)$ be the number of simsun permutations in $\mathfrak{S}_n$ with k descents. By inserting $n+1$ into a simsun permutation in $\mathfrak{S}_n$, establish the recurrence

$$f_k(n+1) = (n - 2k + 2)f_{k-1} + (k+1)f_k(n),$$

with the initial conditions $f_0(1) = 1$, $f_k(n) = 0$ for $k > \lfloor n/2 \rfloor$. Further details may be found in S. Sundaram, *Advances in Math.* **104** (1994), 225–296 (§3) in the context of symmetric functions. We can also give a bijective proof, as follows. Let $\mathcal{E}$ be a flip equivalence class of binary trees on the vertex set $[n+1]$. There are E_{n+1} such flip equivalence classes. (Proposition 1.6.2) . There is a unique tree $T' \in \mathcal{E}$ such that

(i) the path from the root 1 to $n+1$ moves to the right, (ii) for every vertex not on this path with two children, the largest child is on the left, and (iii) any vertex with just one child has this child on the right. Let $w' \in \mathfrak{S}_{n+1}$ satisfy $T' = T(w')$ (as in Section 1.5). Then w' ends in $n+1$; let $w \in \mathfrak{S}_n$ be w' with $n+1$ removed. It is not hard to check that the map $\mathcal{E} \mapsto w$ gives a bijection between flip equivalence classes and simsun permutations. This proof is due to Maria Monks (October 2007).

Simsun permutations are named after Rodica Simion and Sheila Sundaram. They first appear in the paper S. Sundaram, *ibid.* (p. 267). They are variants of the *André permutations* of Foata and Schützenberger [1.27]. The terminology "simsun permutation" is due to S. Sundaram (after they were originally called "Sundaram permutations" by R. Stanley) in *J. Algebraic Combin.* **4** (1995), 69–92 (p. 75). For some further work on simsun permutations, see G. Hetyei, *Discrete Comput. Geom.* **16** (1996), 259–275.

141. (a) *Hint.* Show that $E_{n+1,k}$ is the number of alternating permutations of $[n+2]$ with first term $k+1$ and second term unequal to k, and that $E_{n,n-k}$ is the number of alternating permutations of $[n+2]$ with first term $k+1$ and second term k.

The numbers $E_{n,k}$ are called *Entringer numbers*, after R. C. Entringer, *Nieuw. Arch. Wisk.* **14** (1966), 241–246. The triangular array (1.140) is due to L. Seidel, *Sitzungsber. Münch. Akad.* **4** (1877), 157–187 (who used the word "boustrophedon" to describe the triangle). It was rediscovered by A. Kempner, *Tôhoku Math. J.* **37** (1933), 347–362; R. C. Entringer, *op. cit.*; and V. I. Arnold, *Duke Math. J.* **63** (1991), 537–555. For further information and references, see J. Millar, N. J. A. Sloane, and N. E. Young, *J. Combinatorial Theory, Ser. A* **76** (1996), 44–54. A more recent reference is R. Ehrenborg and S. Mahajan, *Ann. Comb.* **2** (1998), 111–129 (§2). The boustrephedon triangle was generalized to permutations with an arbitrary descent set by Viennot [1.75].

(b) Rotate the triangle and change the sign of E_{mn} when $m+n \equiv 1, 2 \pmod 4$ to obtain the array

$$\begin{array}{ccccccccccccc}
1 & & 0 & & -1 & & 0 & & 5 & & 0 & & \cdots \\
 & -1 & & -1 & & 1 & & 5 & & -5 & & & \\
 & & 0 & & 2 & & 4 & & -10 & & & & \\
 & & & 2 & & 2 & & -14 & & & & & \\
 & & & & 0 & & -16 & & & & & & \\
 & & & & & -16 & & \cdots & & & & & \\
 & & & & & & \ddots & & & & & &
\end{array}$$

This array is just a difference table, as defined in Section 1.9. By (a) the exponential generating function for the first row is $\sec(ix) = \operatorname{sech}(x)$. By Exercise 1.154(c) we get

$$\sum_{m\geq 0}\sum_{n\geq 0}(-1)^{\lfloor (2m+2n+3)/4\rfloor} E_{m+n,[m,n]}\frac{x^m}{m!}\frac{y^n}{n!} = e^{-x}\operatorname{sech}(x+y).$$

If we convert all the negative coefficients to positive, it's not hard to see that the generating function becomes the right-hand side of equation (1.141), as claimed.

The transformation into a difference table that we have used here appears in Seidel, *op. cit.*, and is treated systematically by D. Dumont, *Sém. Lotharingien de Combinatoire* **5** (1981), B05c (electronic). Equation (1.141) appears explicitly in R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, second ed., Addison-Wesley, Reading, MA, 1994 (Exercise 6.75).

142. It is easy to verify that

$$\sum_{n\geq 0} f_n(a)x^n = (\sec x)(\cos(a-1)x + \sin ax),$$

and the proof follows. The motivation for this problem comes from the fact that for $0 \leq a \leq 1$, $f_n(a)$ is the volume of the convex polytope in $\mathbb{R}^n$ given by

$$x_i \geq 0 \ (1 \leq i \leq n), \quad x_1 \leq a, \quad x_i + x_{i+1} \leq 1 \ (1 \leq i \leq n-1).$$

For further information on the case $a = 1$, see Exercise 4.56(c).

143. (a) *Combinatorial proof.* Let $1 \leq i \leq n$. The number of permutations $w \in \mathfrak{S}_n$ fixing i is $(n-1)!$. Hence the total number of fixed points of all $w \in \mathfrak{S}_n$ is $n\cdot(n-1)! = n!$.
Generating function proof. We have

$$f(n) := \sum_{w\in\mathfrak{S}_n} \operatorname{fix}(w) = n!\frac{d}{dt_1} Z_n|_{t_i=1},$$

where Z_n is defined by (1.25). Hence by Theorem 1.3.3 we get

$$\begin{aligned}\sum_{n\geq 0} f(n)\frac{x^n}{n!} &= \frac{d}{dt_1}\exp\left(t_1x + t_2\frac{x^2}{2} + t_3\frac{x^3}{3} + \cdots\right)\Bigg|_{t_i=1} \\ &= x\exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \\ &= \frac{x}{1-x},\end{aligned}$$

whence $f(n) = n!$.
Algebraic proof. Let G be a finite group acting a set Y. By Burnside's lemma (Lemma 7.24.5), also called the Cauchy-Frobenius lemma, the average number of fixed points of $w \in G$ is the number of orbits of the action. Since the "defining representation" of $\mathfrak{S}_n$ on $[n]$ has one orbit, the proof follows.

(b) This result is a straightforward consequence of Proposition 6.1 of R. Stanley, *J. Combinatorial Theory, Ser. A* **114** (2007), 436–460. Is there a combinatorial proof?

144. (a) It is in fact not hard to see that

$$2q^n\frac{\prod_{j=1}^{n}(1-q^{2j-1})}{\prod_{j=1}^{2n+1}(1+q^j)} = \frac{2(2n-1)!!}{3^n}x^n + O(x^{n+1}),$$

where $(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1)$.

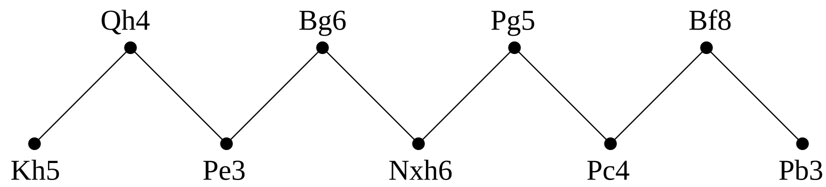


Figure 1.35: The solution poset for Exercise 1.145

(b) See page 450 of R. Stanley, *J. Combinatorial Theory, Ser. A* **114** (2007), 436–460.

145. One solution is 1.Kh5 2.Pe3 3.Nxh6 4.Pc4 5.Pb3 6.Qh4 7.Bg6 8.Rg5 9.Bf8, followed by Nf6 mate. Label these nine Black moves as 1,3,5,7,9,2,4,6,8 in the order given. All solutions are a permutation of the nine moves above. If $a_1, a_2, \dots, a_9$ is a permutation w of the *labels* of the moves, then they correspond to a solution if and only w^{-1} is reverse alternating. (In other words, Qh4 must occur after both Kh5 and Pe3, Bg6 must occur after both Pe3 and Nxh6, etc.). In the terminology of Chapter 3, the solutions correspond to the linear extensions of the "zigzag poset" shown in Figure 1.35. Hence the number of solutions is $E_9 = 7936$. For some properties of zigzag posets, see Exercise 3.66.

146. The proof is a straightforward generalization of the proof we indicated of equation (1.59). For a q-analogue, see Proposition 3.3.19.4 and the discussion following it.

147. A binary tree is an unlabelled min-max tree if and only if every non-endpoint vertex has a nonempty left subtree. Let f_n be the number of such trees on n vertices. Then

$$f_{n+1} = \sum_{k=1}^{n} f_k f_{n-k}, \quad n \geq 1.$$

Setting $y = \sum_{n\geq 0} f_n x^n$ we obtain

$$\frac{y-1-x}{x} = y^2 - y.$$

It follows that

$$y = \frac{1+x-\sqrt{1-2x-3x^2}}{2x}.$$

Comparing with the definition of M_n in Exercise 6.27 shows that $f_n = M_{n-1}$, $n \geq 1$.

148. It is easy to see from equations (1.33) and (1.63) that

$$\Psi_n(a+b, a) = \sum_{S\subseteq[n-1]} \alpha(S) u_S.$$

The proof follows from the formula $\Psi(a,b) = \Phi(a+b, ab+ba)$ (Theorem 1.6.3).

149. *Hint.* First establish the recurrence

$$2\Phi_n = \sum_{\substack{0<i<n \\ n-i=2j-1}} \binom{n}{i} \Phi_i c(c^2-2d)^{j-1} - \sum_{\substack{0<i<n \\ n-i=2j}} \binom{n}{i} \Phi_i (c^2-2d)^j$$

$$+ \begin{cases} 2(c^2-2d)^{k-1}, & n = 2k-1 \\ 0, & n = 2k. \end{cases}$$

The generating function follows easily from multiplying this recurrence by $x^n/n!$ and summing on $n \geq 1$.

This result is due to R. Stanley, *Math. Z.* **216** (1994), 483–499 (Corollary 1.4).

151. This elegant result is due to R. Ehrenborg, private communication (2007), based on the Pyr operator of R. Ehrenborg and M. Readdy, *J. Algebraic Combin.* **8** (1998), 273–299. Using concepts from Chapter 3, the present exercise has the following interpretation. Let P be the poset whose elements are all cd-monomials. Define α to cover β in P if β is obtained from α by removing a c or changing a d to c. Then $[\mu]\Phi_n(c,d)$ is equal to the number of maximal chains of the interval $[1,\mu]$. The problem of counting such chains was considered by F. Bergeron, M. Bousquet-Mélou, and S. Dulucq, *Ann. Sci. Math. Québec* **19** (1995), 139–151. They showed that the total number of saturated chains from 1 to rank n is E_{n+1} (the sum of the coefficients of Φ_{n+1}), though they did not interpret the number of maximal chains in each interval. Further properties of the poset P (and some generalizations) were given by B. Sagan and V. Vatter, *J. Algebraic Combin.* **24** (2006), 117–136.

152. An analogous result where simsum permutations are replaced by "André permutations" was earlier proved by M. Purtill [1.65]. The result for simsun permutations was stated without proof by R. Stanley, *Math. Zeitschrift*, **216** (1994), 483–499 (p. 498), saying that it can be proved by "similar reasoning" to Purtill's. This assertion was further explicated by G. Hetyei, *Discrete Comput. Geom.* **16** (1996), 259–275 (Remark on p. 270).

153. (a) *First solution.* Put $c = 0$ and $d = 1$ in equation (1.142) (so $a - b = \sqrt{-2}$) and simplify. We obtain

$$\sum_{n \geq 0} f(n) \frac{x^{2n+1}}{(2n+1)!} = \sqrt{2}\tan(x/\sqrt{2}).$$

The proof follows from Proposition 1.6.1.

Second solution. By equations (1.62) and (1.64) we have that $2^n f(n)$ is the number of complete (i.e., every internal vertex has two children) min-max trees with n internal vertices. A complete min-max tree with $n+1$ internal vertices is obtained by placing either 1 or $2n+3$ at the root, forming a left complete min-max subtree whose vertices are $2k+1$ elements from $\{2, 3, \ldots, 2n+2\}$ $(0 \leq k \leq n)$, and forming a right complete min-max subtree with the remaining elements. Hence setting $g(n) = 2^n f(n)$ we obtain the recurrence

$$g(n+1) = 2\sum_{k=0}^{n} \binom{2n+1}{2k+1} g(k)g(n-k).$$

It is then straightforward to show that $g(n) = E_{2n+1}$. The result of this exercise was first proved by Foata and Schützenberger [1.27, Propriété 2.6] in the context of André polynomials.

R. Ehrenborg (private communication, 2007) points out that there is a similar formula for the coefficient of any monomial in Φ_n not containing two consecutive c's.

(b) See R. L. Graham and N. Zang, Enumerating split-pair arrangements, preprint dated January 10, 2007. For some further combinatorial interpretations of F_n, see C. Poupard, *European J. Combinatorics* **10** (1989), 369–374; A. G. Kuznetsov, I. M. Pak and A. E. Postnikov, *Uspekhi Mat. Nauk* **49** (1994), 79–110; and M. P. Develin and S. P. Sullivant, *Ann. Combinatorics* **7** (2003), 441–466 (Corollary 5.7).

154. (a) Use equation (1.98).

(b) By (a), $e^{-x}F'(x) = F(x)$, from which $F(x) = e^{e^x - 1}$, so $f(n)$ is the Bell number $B(n)$. The difference table in question looks like

$$\begin{array}{ccccccccccccc}
1 & & 2 & & 5 & & 15 & & 52 & & 203 & & \cdots \\
 & 1 & & 3 & & 10 & & 37 & & 151 & & \cdots & \\
 & & 2 & & 7 & & 27 & & 114 & & \cdots & & \\
 & & & 5 & & 20 & & 87 & & \cdots & & & \\
 & & & & 15 & & 67 & & \cdots & & & & \\
 & & & & & 52 & & \cdots & & & & & \\
 & & & & & & \ddots & & & & & &
\end{array}$$

Note that the first row is identical to the leftmost diagonal below the first row. This "Bell number triangle" is due to C. S. Peirce, *Amer. J. Math.* **3** (1880), 15–57 (p. 48). It gained some popularity by appearing in the "Mathematical Games" column of M. Gardner [1.33, Fig. 13]. D. E. Knuth uses it to develop properties of Bell numbers in *The Art of Computer Programming*, vol. 4, Fascicle 3, Addison-Wesley, Upper Saddle River, NJ, 2005 (Section 7.2.1.5) and gives some further properties in Exercises 7.2.1.5–26 to 7.2.1.5–31.

(c) By Taylor's theorem and (a) we have

$$\begin{aligned}
\sum_{n\geq 0}\sum_{k\geq 0} \Delta^n f(k) \frac{x^n}{n!}\frac{t^k}{k!} &= e^{-x}\left(F(x) + F'(x)t + F''(x)\frac{t^2}{2!} + \cdots\right) \\
&= e^{-x}F(x+t).
\end{aligned}$$

This result appears in D. Dumont and X. G. Viennot, *Ann. Discrete Math.* **6** (1980), 77–87, but is undoubtedly much older.

155. (a) For further information related to this problem and Exercise 1.154(a), see D. Dumont, in *Séminaire Lotharingien de Combinatoire*, 5ème Session, Institut de Recherche Mathématique Avancée, Strasbourg, 1982, pp. 59–78.

(b) One computes $f(0) = 1$, $f(1) = 2$, $f(2) = 6$, $f(3) = 20, \ldots$. Hence guess $f(n) = \binom{2n}{n}$ and $F(x) := \sum f(n)x^n = (1-4x)^{-1/2}$. By (a) we then have $G(x) := \sum g(n)x^n = \frac{1}{1+x}F\left(\frac{x}{1+x}\right) = (1-2x-3x^2)^{-1/2}$. To verify the guess, one must check that $\frac{1}{1+x}G\left(\frac{x}{1+x}\right) = F(x^2)$, which is routine.

(c) (suggested by L. W. Shapiro) One computes $f(0) = 1$, $f(1) = 1$, $f(2) = 2$, $f(3) = 5$ $f(4) = 14, \ldots$. Hence guess $f(n) = \frac{1}{n+1}\binom{2n}{n}$ (the Catalan number C_n) and $F(x) := \sum f(n)x^n = \frac{1}{2x}(1-(1-4x)^{1/2})$. Then

$$F_1(x) := \sum f(n+1)x^n = \frac{1}{x}(F(x)-1) = \frac{1}{2x^2}(1-2x-(1-4x)^{1/2}),$$

so by (a),

$$G(x) := \sum g(n)x^n = \frac{1}{1+x}F_1\left(\frac{1}{1+x}\right) = \frac{1}{2x^2}(1-x-(1-2x-3x^2)^{-1/2}).$$

To verify this guess, one must check that $\frac{1}{1+x}G\left(\frac{1}{1+x}\right) = F(x^2)$, which is routine.

156. *Answer:* $c_n = \prod_p p^{\lfloor n/p \rfloor}$, where p ranges over all primes. Thus $c_0 = 1$, $c_1 = 1$, $c_2 = 2$, $c_3 = 6$, $c_4 = 12$, $c_5 = 60$, $c_6 = 360$, and so on. See E. G. Strauss, *Proc. Amer. Math. Soc.* **2** (1951), 24–27. The sequence c_n can also be defined by the recurrence $c_0 = 1$ and $c_{n+1} = s_{n+1}c_n$, where s_{n+1} is the largest squarefree divisor of $n+1$.

157. Let $z = y^\lambda$, and equate coefficients of x^{n-1} on both sides of $(\lambda+1)y'z = (yz)'$. This result goes back to Euler and is discussed (with many similar methods for manipulating power series) in D. E. Knuth, *The Art of Computer Programming*, vol. 2, third ed., Addison-Wesley, Upper Saddle River, NJ, 1997 (Section 4.7). It was rediscovered by H. W. Gould, *Amer. Math. Monthly* **81** (1974), 3–14.

158. Let $\log F(x) = \sum_{n\geq 1} g_n x^n$. Then

$$\sum_{n\geq 1} g_n x^n = \sum_{i\geq 1}\sum_{j\geq 1} \frac{a_i x^{ij}}{j} = \sum_{n\geq 1} \frac{x^n}{n} \sum_{d|n} d a_d.$$

Hence

$$n g_n = \sum_{d|n} d a_d,$$

so by the Möbius inversion formula of elementary number theory,

$$a_n = \frac{1}{n}\sum_{d|n} d g_d \mu(n/d). \tag{1.156}$$

We have $1 + x = (1-x)^{-1}(1-x^2)$ (no need to use (1.156)).

If $F(x) = e^{x/(1-x)}$ then $g_n = 1$ for all n, so by (1.156) we have $a_n = \phi(n)/n$, where $\phi(n)$ is Euler's totient function.

159. *Answer:* $A(x) = \sqrt{F(x)F(-x)}$, $B(x) = \sqrt{F(x)/F(-x)}$. This result is due to Marcelo Aguiar (private communication, 2006) as part of his theory of combinatorial Hopf algebras and noncommutative diagonalization.

160. (a) This formula is a standard result of hoary provenance which follows readily from

$$\sum_{r=0}^{k-1} \zeta^{rj} = \begin{cases} 0, & 0 < j < k \\ k, & j = 0. \end{cases}$$

(b) Let $\zeta = e^{2\pi i/k}$. According to (a) and Proposition 1.4.6 we have

$$f(n,k,j) = \frac{1}{k} \sum_{r=0}^{k-1} \zeta^{-jr} (\boldsymbol{n})!\big|_{q=\zeta^r}. \tag{1.157}$$

If $n \geq k$ then at least one factor $1 + q + \cdots + q^m$ of $(\boldsymbol{n})!$ will vanish at $q = \zeta^r$ for $1 \leq r \leq k-1$. Thus the only surviving term of the sum is $(\boldsymbol{n})!\big|_{q=1} = n!$, and the proof follows.

(c) When $n = k-1$, we have $(\boldsymbol{n})!\big|_{q=\zeta^r} = 0$ unless $r = 0$ or ζ^r is a primitive kth root of unity. In the former case we get the term $(k-1)!/k$. In the latter case write $\xi = \zeta^r$. Then

$$(\boldsymbol{k-1})!\big|_{q=\xi} = \frac{(1-\xi)(1-\xi^2)\cdots(1-\xi^{k-1})}{(1-\xi)^{k-1}}. \tag{1.158}$$

Now

$$\prod_{j=1}^{k-1} (q - \xi^j) = \frac{q^k - 1}{q - 1},$$

Letting $q \to 1$ gives $\prod_{j=1}^{k-1}(1-\xi^j) = k$. Hence from equation (1.158) we have

$$(\boldsymbol{k-1})!\big|_{q=\xi} = \frac{k}{(1-\xi)^{k-1}},$$

and the proof follows from setting $n = k-1$ and $j = 0$ in equation (1.157).

NOTE. Let $\Phi_n(x)$ denote the (monic) nth cyclotomic polynomial, i.e., its zeros are the primitive nth roots of unity. It can be shown that if $n \geq 2$ then

$$f(n-1,n,0) = \frac{(n-1)!}{n} + (-1)^n (n-1)[x^{n-1}] \log \frac{\Phi_n(1+x)}{\Phi_n(1)}.$$

Let us also note that $\Phi_n(1) = p$ if n is the power of a prime p; otherwise $\Phi_n(1) = 1$.

161. (b) We have

$$\frac{H(x)}{H(x) + H(-x)} = \frac{G(x)}{2}.$$

Hence

$$\begin{aligned}\frac{H(-x)}{H(x)} &= \frac{2}{G(x)} - 1 \\ \Rightarrow \log H(-x) - \log H(x) &= \log\left(\frac{2}{G(x)} - 1\right).\end{aligned}$$

If we divide the left-hand side by -2 then we obtain the odd part of $\log H(x)$. Hence

$$\log H(x) = -\frac{1}{2}\log\left(\frac{2}{G(x)} - 1\right) + E_1(x),$$

where $E_1(x)$ is any even power series in x with $E_1(0) = 0$. Thus $E(x) := e^{E_1(x)}$ is an arbitrary even power series with $E(0) = 1$. Therefore we get the general solution

$$H(x) = \left(\frac{2}{G(x)} - 1\right)^{-1/2} E(x).$$

162. Using the formulas

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)},$$

$$\tan x/2 = \frac{\pm\sqrt{1+\tan^2 x} - 1}{\tan x},$$

we have $\tan(\tan^{-1} f(x) + \tan^{-1} f(-x)) = g(x)$

$$\Rightarrow \tan^{-1} f(x) = \frac{1}{2}\tan^{-1} g(x) + k(x), \quad k(x) = -k(-x)$$

$$\begin{aligned}\Rightarrow f(x) &= \tan\left(\frac{1}{2}\tan^{-1} g(x) + k(x)\right) \\ &= \frac{\tan\frac{1}{2}\tan^{-1} g(x) + \tan k(x)}{1 - (\tan\frac{1}{2}\tan^{-1} g(x))\tan k(x)} \\ &= \frac{\frac{\sqrt{1+g(x)^2}-1}{g(x)} + h(x)}{1 - \frac{\sqrt{1+g(x)^2}-1}{g(x)}h(x)}, \quad h(x) = -h(-x)\end{aligned}$$

Choosing the correct sign gives

$$f(x) = \frac{-\sqrt{1+g(x)^2} - 1 + g(x)h(x)}{g(x) - (\sqrt{1+g(x)^2} - 1)h(x)},$$

where $h(x)$ is any even power series.

163. (a) We have $F(x,y) = f(f^{\langle -1\rangle}(x) + f^{\langle -1\rangle}(y))$. The concept of a formal group law goes back to S. Bocher, *Ann. Math.* **47** (1946), 192–201.

(b) See for instance A. Fröhlich, *Lecture Notes in Math.*, no. 74, Springer-Verlag, Berlin/New York, 1968. For a combinatorial approach to formal groups via Hopf algebras, see C. Lenart, Ph.D. thesis, University of Manchester, 1996, and C. Lenart and N. Ray, Some applications of incidence Hopf algebras to formal group theory and algebraic topology, preprint, University of Manchester, 1995.

(c) $f(x) = x, e^x - 1, \tan x, \sin x$, respectively.

(d) Let $R(x) = (xe^{-x})^{\langle -1 \rangle}$. Thus

$$\begin{aligned} F(x,y) &= (R(x)+R(y))e^{-R(x)-R(y)} \\ &= xe^{-R(y)} + ye^{-R(x)}. \end{aligned}$$

The proof follows from equation (5.128), which asserts that

$$e^{-R(x)} = 1 - \sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}.$$

(e) Euler, Institutiones Calculi integralis, *Ac. Sc. Petropoli*, 1761, showed that

$$F(x,y) = \frac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1+x^2y^2}.$$

164. Note that setting $x = 0$ is useless. Instead write

$$F(x,y) = \frac{xF(x,0) - y}{xy^2 + x - y}.$$

The denominator factors as $x(y-\theta_1(x))(y-\theta_2(x))$, where

$$\theta_1(x), \theta_2(x) = \frac{1 \mp \sqrt{1-4x^2}}{2x}.$$

Now $y - \theta_1(x) \sim y - x$ as $x, y \to 0$, so the factor $1/(y-\theta_1(x))$ has no power series expansion about $(0,0)$. Since $F(x,y)$ has such an expansion, the factor $y - \theta_1(x)$ must appear in the numerator. Hence $xF(x,0) = \theta_1(x)$, yielding

$$\begin{aligned} F(x,0) &= \frac{1-\sqrt{1-4x^2}}{2x^2} = \sum_{n\geq 0} C_n x^{2n} \\ F(x,y) &= \frac{2}{1-2xy+\sqrt{1-4x^2}}. \end{aligned}$$

The solution to this exercise is a simple example of a technique known as the *kernel method*. This method originated in Exercise 2.2.1-.4 of Knuth's book *The Art of Computer Programming*, vol. 1, Addison-Wesley, 1973, third edition, 1997. The present exercise is the same as Knuth's (after omitting some preliminary steps). See Section 1 of H. Prodinger, *Sém. Lotharingien de Combinatoire* **50** (2004), article B50f, for further information and examples. An interesting variant of the kernel method applied to queuing theory appears in Chapter 14 of L. Flatto, *Poncelet's Theorem*, American Math. Society, Providence, RI, 2009.

165. *Answer:* the coefficient $f(n)$ of $F(x)$ is the number of 1's in the binary expansion of n.

166. *Answer:* $F(x) = (1+x^n)^{1/n} = \sum_{k\geq 0} \binom{1/n}{k} x^{kn}$.

167. Equation (1.143) is just the Taylor series expansion of $F(x+t)$ at $t=0$.

168. (a) It is not hard to check that for general $A(x) = x + a_2x^2 + a_3x^3 + \cdots$, we have

$$A(-A(-x)) = x + p_2x^2 + p_3x^3 + \cdots ,$$

where p_{2n-1} and p_{2n} are polynomials in $a_2, a_3, \dots, a_{2n-1}$. (It's easy to see that in fact $p_2 = 0$.) Moreover, the only term of p_{2n-1} involving a_{2n-1} is $2a_{2n-1}$. Hence if $A(-A(-x)) = x$ then once $a_2, a_3, \dots, a_{2n-2}$ are specified, we have that a_{2n-1} is uniquely determined. Thus we need to show that if $a_2, a_4, ..., a_{2n-2}$ are specified, thereby determining $a_3, a_5, \dots, a_{2n-1}$, then $p_{2n} = 0$. For instance, equating coefficients of x^3 in $A(-A(-x)) = x$ gives $a_3 = a_2^2$. Then

$$p_4 = a_2^3 - a_2a_3 = a_2^3 - a_2(a_2^2) = 0.$$

We can reformulate the result we need to prove more algebraically. Given $A(x) = x + a_2x^2 + \cdots$, let $B(x) = A(-A(-x)) = x + p_2x^2 + \cdots$. Then we need to show that $p_{2n} \in I := \langle p_2, \dots, p_{2n-1} \rangle$, the ideal of the polynomial ring $K[a_2, a_3, \dots]$ generated by $p_1, \dots, p_{2n-1}$.
Let $A^{\langle -1 \rangle}(x) = x + \alpha_2x^2 + \alpha_3x^3 + \cdots$. Then

$$A(-x) = B(-A^{\langle -1 \rangle}(x)) = A^{\langle -1 \rangle}(x) + p_2A^{\langle -1 \rangle}(x)^2 - \cdots .$$

Taking the coefficient of x^{2n} gives

$$a_{2n} \equiv -\alpha_{2n} + p_{2n} \pmod{I}. \tag{1.159}$$

But also

$$\begin{aligned} -A(-x) &= A^{\langle -1 \rangle}(B(x)) \\ &= B(x) + \alpha_2B(x)^2 + \cdots . \end{aligned}$$

Taking coefficients of x^{2n} yields

$$\begin{aligned} -a_{2n} &\equiv p_{2n} + [x^{2n}] \sum_{i=2}^{2n} \alpha_i(x + p_{2n}x^{2n})^i \pmod{I} \\ &\equiv p_{2n} + \alpha_{2n} \pmod{I}. \end{aligned} \tag{1.160}$$

Equations (1.159) and (1.160) imply $p_{2n} \in I$, as desired. This proof was obtained in collaboration with Whan Ghang.

NOTE. It was shown by Ghang that a_{2n+1} is a polynomial in $a_2, a_4, \dots, a_{2n}$ with *integer* coefficients.

NOTE. An equivalent reformulation of the result of this item is the following. For any $A(x) = x + a_2x^2 + \cdots \in K[[x]]$, either $A(-A(-x)) = x$ or $A(-A(-x)) - x$ has odd degree. This result can be considerably generalized. For instance, if $C(x) = -x + c_2x^2 + \cdots$ and $C(C(x)) = x$, then (writing composition of functions as juxtaposition) either $ACAC(x) = x$ or $ACAC(x) - x$ has odd degree. More generally, if ζ is a primitive kth root of unity and $C(x) = \zeta x + c_2x^2 + \cdots$, where $C^k = x$, then either $(AC)^k(x) = x$ or $(AC)^k(x) - x$ has degree $d \equiv 1 \pmod k$. The possibility of such a generalization was suggested by F. Bergeron (private communication, 2007).

(d) Use induction on n.

(f) Marcelo Aguiar (private communication, 2006) first obtained this result as part of his theory of combinatorial Hopf algebras and noncommutative diagonalization.

(g) *Answer:* $A(x) = 2x/(2-x)$ and $D(x) = \log \frac{2+x}{2-x}$. This example is due to Aguiar.

(i) First show the following.

- $\displaystyle\sum_{n\geq 1} a_n \left(\frac{x}{1-x}\right)^n = \sum_{n\geq b_n} x^n \iff e^x \sum_{j\geq 0} a_{j+1}\frac{x^j}{j!} = \sum_{j\geq 0} b_{j+1}\frac{x^j}{j!}.$
  (See Exercises 154(a) and 155(a).)
- For any $F(x) = x + \sum_{n\geq 2} a_nx^n$ and $H(x) = x + \sum_{n\geq 2} b_nx^n$, we have

$$F^{\langle -1\rangle}(-F(-x)) = H^{\langle -1\rangle}(-H(-x))$$

  if and only if $F(x)/H(x)$ is odd.

(j) *Answer.* $b_{2n} = (-1)^{n-1}E_{2n-1}$, where E_{2n-1} is an Euler number.

169. There are many possible methods. A uniform way to do all three parts is to note that for any power series $F(x) = \sum_{n\geq} a_nx^n$, we have

$$xDF(x) = \sum_{n\geq 0} na_nx^n,$$

where $D = \frac{d}{dx}$. Hence

$$(xD+2)^2F(x) = \sum_{n\geq 0}(n+2)^2a_nx^n.$$

Letting $F(x) = 1/(1-x), e^x$, and $1/\sqrt{1-4x}$ yields after some routine computation the three answers

$$\begin{aligned}
\sum_{n\geq 0}(n+2)^2x^n &= \frac{4-3x+x^2}{(1-x)^3}\\
\sum_{n\geq 0}(n+2)^2\frac{x^n}{n!} &= (x^2+5x+4)e^x\\
\sum_{n\geq 0}(n+2)^2\binom{2n}{n}x^n &= \frac{4-22x+36x^2}{(1-4x)^{3/2}}.
\end{aligned}$$

170. (a) *Answer:* $y = (\alpha + (\beta - \alpha)x)/(1 - x - x^2)$. The general theory of linear recurrence relations with constant coefficients is developed in Sections 4.1–4.4.

(b) The recurrence yields $y' = (xy)' - \frac{1}{2}xy^2$, $y(0) = 1$, from which we obtain

$$y = \frac{\exp\left(\frac{x}{2} + \frac{x^2}{4}\right)}{\sqrt{1-x}}.$$

For the significance of this generating function, see Example 5.2.9.

(c) We obtain $2y' = y^2$, $y(0) = 1$, whence $y = 1/(1 - \frac{1}{2}x)$. Thus $a_n = 2^{-n}n!$.

(d) (sketch) Let $F_k(x) = \sum_{n\geq 0} a_k(n)x^n/n!$, so $A(x,t) = \sum_{k\geq 0} F_k(x)t^k$. The recurrence for $a_k(n)$ gives

$$F_k'(x) = \sum_{2r+s=k-1} (F_{2r}(x) + F_{2r+1}(x))F_s(x). \tag{1.161}$$

Let $A_e(x) = \frac{1}{2}(A(x,t) + A(x,-t))$ and $A_o(x,t) = \frac{1}{2}(A(x,t) - A(x,-t))$. From equation (1.161) and some manipulations we obtain the system of differential equations

$$\begin{aligned} \frac{\partial A_e}{\partial x} &= tA_eA_o + A_o^2 \\ \frac{\partial A_o}{\partial x} &= tA_e^2 + A_eA_o. \end{aligned} \tag{1.162}$$

To solve this system, note that

$$\frac{\partial A_e/\partial x}{\partial A_o/\partial x} = \frac{A_o}{A_e}.$$

Hence $\frac{\partial}{\partial x}(A_e^2 - A_o^2) = 0$, so $A_e^2 - A_o^2$ is independent of x. Some experimentation suggests that $A_e^2 - A_o^2 = 1$, which together with (1.162) yields

$$\frac{\partial A_e}{\partial x} = tA_e\sqrt{A_e^2 - 1} + A_e^2 - 1.$$

This equation can be routinely solved by separation of variables (though some care must be taken to choose the correct branch of the resulting integral, including the correct sign of $\sqrt{A_e^2 - 1}$). A similar argument yields A_o, and we finally obtain the following expression for $A = A_e + A_o$:

$$A(x,t) = \sqrt{\frac{1-t}{1+t}}\left(\frac{2}{1 - \frac{1-\rho}{t}e^{\rho x}} - 1\right),$$

where $\rho = \sqrt{1-t^2}$. It can then be checked that this formula does indeed give the correct solution to the original differential equations, justifying the assumption that $A_e^2 - A_o^2 = 1$. For further details and motivation, see Section 2 of R. Stanley, *Michigan Math. J.*, to appear; `arXiv:math/0511419`.

171. While this problem can be solved by the "brute force" method of computing the coefficients on the right-hand side of equation (1.144), it is better to note that $B'(x)-B(x) = A'(x)$ and then solve this differential equation for $B(x)$ with the initial condition $B(0) = a_0$. Alternatively, one could start with $A(x)$:

$$B(x) = (1 + I + I^2 + \cdots)A(x) = (1 - I)^{-1}A(x).$$

Multiplying by $1 - I$ and differentiating both sides results in the same differential equation $B'(x) - B(x) = A'(x)$. (It isn't difficult to justify these formal manipulations of the operator I.)

172. One method of proof is to first establish the three term recurrence

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$$

and then use induction.

173. (a)

$$\begin{aligned}\sqrt{\frac{1+x}{1-x}} &= (1 + x)(1 - x^2)^{-1/2}\\ &= \sum_{n\geq 0} 4^{-n}\binom{2n}{n}(x^{2n} + x^{2n+1})\end{aligned}$$

(b) $\displaystyle\sum_{n\geq 1} \frac{x^{2n}}{n^2\binom{2n}{n}}$

(c) $\displaystyle\sum_{n\geq 0} t(t^2 - 1^2)(t^2 - 3^2)\cdots(t^2 - (2n - 1)^2)\frac{x^{2n+1}}{(2n + 1)!}$

(d) $\displaystyle\sum_{n\geq 0} t^2(t^2 - 2^2)(t^2 - 4^2)\cdots(t^2 - (2n - 2)^2)\frac{x^{2n}}{(2n)!}$

(e) $\displaystyle 2\sum_{n\geq 0}(-1)^n 2^{2n}\frac{x^{4n+2}}{(4n + 2)!}$. Similar results hold for $\cos(x)\cosh(x)$, $\cos(x)\sinh(x)$, and $\sin(x)\cosh(x)$.

(f) $\displaystyle 6\sum_{n\geq 0}(-1)^n 2^{6n}\frac{x^{6n+3}}{(6n + 3)!}$. Similar results hold when any subset of the three sin's is replaced by cos. There seems to be no analogous result for *four* factors.

(g) $\Re\binom{i}{n}$, where $i^2 = -1$

To do (c), for instance, first observe that the coefficient of $x^{2n+1}/(2n + 1)!$ in $\sin(t\sin^{-1} x)$ is a polynomial $P_n(t)$ of degree $2n+1$ and leading coefficient $(-1)^n$. If $k \in \mathbb{Z}$, then $\sin(2k + 1)\theta$ is an odd polynomial in $\sin\theta$ of degree $2k + 1$. Hence $P_n(\pm(2k + 1)) = 0$ for $n > k$. Moreover, $\sin 0 = 0$ so $P_n(0) = 0$. We now have sufficient information to determine $P_n(t)$ uniquely. To get (b), consider the coefficient of t^2 in (d). For (g), note that

$$\cos(\log(1+x)) = \Re(1+x)^i.$$

174. *Hint:* what is the number of elements of the set $\{0,1\}$?

176. Induction on n. We have $E(0)=0$. For $n \geq 1$ choose the first vector v_1 at random. If $v_1 = 0$, we expect $E(n)$ further steps, and this occurs with probability $1/q^n$. Otherwise v_1 is not the zero vector. Consider the projection of our space to a subspace complementary to v_1. The uniform distribution over $\mathbb{F}_q^n$ projects to the uniform distribution over this copy of $\mathbb{F}_q^{n-1}$, and our sequence of vectors will span $\mathbb{F}_q^n$ precisely when the set of their projections spans $\mathbb{F}_q^{n-1}$. It follows that we expect $E(n-1)$ further steps, and so

$$E(n) = 1 + \frac{E(n) + (q^n - 1)E(n-1)}{q^n}.$$

Solving this equation gives $E(n) = E(n-1) + q^n/(q^n - 1)$, and so

$$E(n) = \sum_{i=1}^{n} q^i/(q^i - 1).$$

This argument was suggested by J. Lewis (October 2009).

182. Suppose that $A \in \mathrm{GL}(n,q)$ has no 0 entries. There are exactly $(q-1)^{2n-1}$ matrices of the form DAD', where D, D' are diagonal matrices in $\mathrm{GL}(n,q)$. Exactly one of the matrices $C = DAD'$ has the first entry in every row and column equal to -1. Subtract the first column of C from every other column, obtaining a matrix D. Let B be obtained from D by removing the first row and column. Then B is a matrix in $\mathrm{GL}(n-1,q)$ with no entry equal to 1, and every such matrix is obtained exactly once by this procedure.

183. *First solution* (sketch). The identity asserts that each of the q^n monic polynomials of degree n can be written uniquely as a product of monic irreducible polynomials.

Second solution (sketch). Take logarithms of both sides and simplify the right-hand side.

185. (b) Note that $q^n - D(n,0)$ is the number of monic polynomials of degree n over $\mathbb{F}_q$ with nonzero discriminant. In the same way that we obtained the first solution to Exercise 1.183, we get

$$\sum_{n\geq 0}(q^n - D(n,0))x^n = \prod_{d\geq 1}(1+x^d)^{\beta(d)}.$$

Hence

$$\begin{aligned}\sum_{n\geq 0}(q^n - D(n,0))x^n &= \left(\frac{1-x^2}{1-x}\right)^{\beta(d)} \\ &= \frac{1-qx^2}{1-qx},\end{aligned}$$

the last step by Exercise 1.183. The proof follows easily. This result appears in D. E. Knuth, *The Art of Computer Programming*, vol. 2, third ed., Addison-Wesley, Reading, MA, 1997 (Exercise 4.6.2-2(b)) and is attributed to E. R. Berlekamp. Greta Panova (November 2007) showed that this problem can also be solved by establishing the recurrence

$$D(n,0)=\sum_{k\geq 1} q^k(q^{2n-k}-D(n-2k,0)).$$

(c) We have

$$\begin{aligned}\sum_{\beta\in\mathbb{N}^k} N(\beta)x^\beta &= \prod_{d\geq 1}\left(\sum_{\alpha\in X} x^{\alpha d}\right)^{\beta(d)}\\ &= \prod_{d\geq 1}\prod_{\substack{\alpha\in\mathbb{N}^k\\ \alpha\neq(0,0,\ldots,0)}} (1-x^{\alpha d})^{a_\alpha \beta(d)}.\end{aligned}$$

The proof follows from Exercise 1.183.

186. (a) Let $k=1$ and $X=\{0,1,\ldots,r-1\}$ in Exercise 1.185(c). We get

$$\sum_{n\in X} x^n = 1+x+\cdots+x^{r-1}=\frac{1-x^r}{1-x}.$$

Hence

$$\sum_{n\geq 0} N_r(n)x^n=\frac{1-qx^r}{1-qx},$$

yielding equation (1.147). This result is stated in D. E. Knuth, *The Art of Computer Programming*, vol. 2, third ed., Addison-Wesley, Reading, MA, 1997 (solution to Exercise 4.6.2-2(b)).

(b) Set $k=2$ and $X=\{(m,0)\,:\,m\in\mathbb{N}\}\cup\{(0,n)\,:\,n\in\mathbb{P}\}$ to get

$$\sum_{m,n\geq 0} N(m,n)x^m y^n=\frac{1-qxy}{(1-qx)(1-qy)},$$

from which equation (1.148) follows.

(c) Take $k=1$ and $X=\mathbb{N}-\{1\}$. We get

$$\sum_{n\geq 0} P(n)x^n=\frac{1-qx^6}{(1-qx^2)(1-qx^3)},$$

via the identity

$$1+\frac{t^2}{1-t}=\frac{1-t^6}{(1-t^2)(1-t^3)}.$$

Using the partial fraction expansion

$$\frac{1-qx^6}{(1-qx^2)(1-qx^3)} = -\frac{x}{q} + \frac{(1+q)(1+x)}{q(1-qx^2)} - \frac{1+qx+qx^2}{q(1-qx^3)},$$

it is routine to obtain equation (1.149). This result can also be obtained by noting that every monic powerful polynomial can be written uniquely in the form f^2g^3, where f and g are monic and g is squarefree. Hence $P(n) = \sum_{2i+3j=n} q^i(q^j - D(j,0))$, where $D(j,0)$ is defined in Exercise 1.185(b), etc. This result is due to R. Stanley (proposer), Problem 11348, *Amer. Math. Monthly* **115** (2008), 262; R. Stong (solution), **117** (2010), 87–88.

NOTE. The term "powerful polynomial" is borrowed from the corresponding notion for integers. See for instance the Wikipedia entry "Powerful number" at

⟨`http://en.wikipedia.org/wiki/Powerful_number`⟩.

187. (a) The *resultant* $\operatorname{res}(f,g)$ of two polynomials $f(x) = \prod(x-\theta_i)$ and $g(x) = \prod(x-\tau_j)$ over a field K is defined by

$$\operatorname{res}(f,g) = \prod_{i,j}(\theta_i - \tau_j).$$

It is a standard fact (a consequence of the fact that $\operatorname{res}(f,g)$ is invariant under any permutation of the θ_i's and of the τ_j's) that $\operatorname{res}(f,g) \in K$. Suppose that $f(x) = f_1(x)\cdots f_k(x)$ where each $f_i(x)$ is irreducible. Clearly

$$\operatorname{disc}(f) = \prod_{i=1}^{k} \operatorname{disc}(f_i) \cdot \prod_{1\leq i<j\leq k} \operatorname{res}(f_i,f_j)^2. \tag{1.163}$$

A standard result from Galois theory states that the discriminant of an irreducible polynomial $g(x)$ of degree n over a field K is a square in K if and only if the Galois group of $g(x)$ (regarded as a group of permutations of the zeros of $g(x)$) is contained in the alternating group $\mathfrak{A}_n$. Now the Galois group of an irreducible polynomial of degree n over $\mathbb{F}_q$ is generated by an n-cycle and hence is contained in $\mathfrak{A}_n$ if and only if n is odd. It follows from equation (1.163) that if $\operatorname{disc}(f) \neq 0$, then $\operatorname{disc}(f)$ is a square in $\mathbb{F}_q$ if and only if $n-k$ is even. This result goes back to L. Stickelberger, *Verh. Ersten Internationaler Mathematiker-Kongresses (Zürich, 1897)*, reprinted by Kraus Reprint Limited, Nendeln/Liechtenstein, 1967, pp. 182–193. A simplification of Stickelberger's argument was given by K. Dalen, *Math. Scand.* **3** (1955), 124–126. See also L. E. Dickson, *Bull. Amer. Math. Soc.* **13** (1906/07), 1–8, and R. G. Swan, *Pacific J. Math.* **12** (1962), 1099–1106 (Corollary 1). The above proof is possibly new. NOTE: Swan, *ibid.* (§3), uses this result to give a simple proof of the law of quadratic reciprocity.

Now let $N_e(n)$ (respectively, $N_o(n)$) denote the number of monic polynomials of degree n which are a product of an even number (respectively, odd number) of distinct irreducible factors. It is easy to see (analogous to the solution to Exercise 1.183) that

$$\sum_{n\geq 0}(N_e(n)-N_o(n))x^n = \prod_{d\geq 1}(1-x^d)^{\beta(d)}.$$

But

$$\prod_{d\geq 1}(1-x^d)^{\beta(d)} = 1-qx$$

by Exercise 1.183. Hence $N_e(n) = N_o(n)$ for $n > 1$, and the proof follows.

(b) Let $f(x) = \prod_{i=1}^n (x-\theta_i)$ be a monic polynomial of degree n over $\mathbb{F}_q$. For $a \in \mathbb{F}_q^* = \mathbb{F}_q - \{0\}$, write $f_a(x) = a^n f(x/a)$, so $f_a(x) = \prod_{i=1}^n (x - a\theta_i)$. It follows that

$$\mathrm{disc}(f_a(x)) = a^{n(n-1)}\mathrm{disc}(f(x)).$$

If $(n(n-1), q-1) = 1$ then the map $a \mapsto a^{n(n-1)}$ is a bijection on $\mathbb{F}_q^*$. Hence if $\mathrm{disc}(f) \neq 0$, then we have $\{\mathrm{disc}(f_a) : a \in \mathbb{F}_q^*\} = \mathbb{F}_q^*$. It follows that $D(n,a) = D(n,b)$ for all $a, b \in \mathbb{F}_q^*$. Since $D(n,0) = q^{n-1}$ we have $D(n,a) = q^{n-1}$ for all $a \in \mathbb{F}_q$.

Now assume that $(n(n-1), q-1) = 2$. Thus as a ranges over $\mathbb{F}_q^*$, $a^{n(n-1)}$ ranges over all squares in $\mathbb{F}_q^*$ twice each. Some care must be taken since we can have $f_a(x) = f_b(x)$ for $a \neq b$. (This issue did not arise in the case $(n(n-1), q-1) = 1$ since the $f_a(x)$'s had distinct discriminants.) Thus for each f let P_f be the *multiset* of all f_a, $a \in \mathbb{F}_q^*$. The multiset union $\bigcup_f P_f$ contains each monic polynomial of degree n over $\mathbb{F}_q$ exactly $q-1$ times. For each $a, b \in \mathbb{F}_q^*$ such that either both a, b or neither a, b are squares, the same number of polynomials (counting multiplicity) $g \in \bigcup_f P_f$ satisfy $\mathrm{disc}(g) = a$ as satisfy $\mathrm{disc}(g) = b$. Finally, by (a) it follows that the number of $g \in \bigcup_f P_f$ with square discriminants is the same as the number with nonsquare discriminants. Hence $D(n,a) = D(n,b)$ for all $a, b \in \mathbb{F}_q^*$, and thus as above for all $a, b \in \mathbb{F}_q$.

188. *First solution.* Let V be an n-dimensional vector space over a field K, and fix an ordered basis $\boldsymbol{v} = (v_1, \dots, v_n)$ of V. Let $\mathcal{N}_n$ denote the set of all nilpotent linear transformations $A : V \to V$. We will construct a bijection $\varphi : \mathcal{N}_n \to V^{n-1}$. Letting $V = \mathbb{F}_q$, it follows that $\#\mathcal{N}_n = \#(\mathbb{F}_q)^{n-1} = q^{n(n-1)}$.

The bijection is based on a standard construction in linear algebra known as *adapting* an ordered basis $\boldsymbol{w} = (w_1, \dots, w_n)$ of a vector space V to an m-dimensional subspace U of V. It constructs from $\boldsymbol{w}$ in a canonical way a new ordered basis $w_{i_1}, \dots, w_{i_{n-m}}, u_1, \dots, u_m$ of V such that the first $n-m$ elements form a subsequence of $\boldsymbol{w}$ and the last m form an ordered basis of u. See e.g. M. C. Crabb, *Finite Fields and Their Applications* **12** (2006), 151–154 (page 153) for further details.

Now let $A \in \mathcal{N}_n$, and write $V_i = A^i(V)$, $i \geq 0$. Let r be the least integer for which $V_r = 0$, so we have a strictly decreasing sequence

$$V = V_0 \supset V_1 \supset \cdots \supset V_r = 0.$$

Set $n_i = \dim V_i$ and $m_i = n_{i-1} - n_i$. Adapt the ordered basis $\boldsymbol{v}$ of V to V_1. Then adapt this new ordered basis to V_2, etc. After $r-1$ steps we have constructed in a canonical way an ordered basis $\boldsymbol{y} = (y_1, \ldots, y_n)$ such that $y_{n-n_i+1}, \ldots, y_n$ is a basis for V_i, $1 \leq i \leq r-1$. We associate with A the $(n-1)$-tuple $\varphi(A) = (A(y_1), \ldots, A(y_{n-1})) \in V^{n-1}$. (Note that $A(y_n) = 0$.) It is straightforward to check that this construction gives a bijection $\varphi : \mathcal{N}_n \to V^{n-1}$ as desired.

This argument is due to M. C. Crabb, *ibid.*, and we have closely followed his presentation (though with fewer details). As Crabb points out, this bijection can be regarded as a generalization of the Prüfer bijection (first proof of Proposition 5.3.2, specialized to rooted trees) for counting rooted trees on an n-element set. Further connections between the enumeration of trees and linear transformations were obtained by J.-B. Yin, Ph.D. thesis, M.I.T., 2009. For a further result of this nature, see Exercise 1.189.

Second solution (sketch), due to Hansheng Diao, November 2007. Induction on n, the base case $n = 1$ being trivial. The statement is true for $k < n$. Let Q be the matrix in $\mathrm{Mat}(n, q)$ with 1's on the diagonal above the main diagonal and 0's elsewhere, i.e., a Jordan block of size n with eigenvalue 0. Let

$$\mathcal{A} = \{(M, N) \in \mathrm{Mat}(n, q) \times \mathrm{Mat}(n, q) \ : \ N \text{ is nilpotent}, \ QM = MN\}.$$

We compute $\#\mathcal{A}$ in two ways. Let $f(n)$ be the number of nilpotent matrices in $\mathrm{Mat}(n, q)$. We can choose N in $f(n)$ ways. Choose $v \in \mathbb{F}_q^n$ in q^n ways. Then there is a unique matrix $M \in \mathrm{Mat}(n, q)$ with first row v such that $QM = MN$. Hence

$$\#\mathcal{A} = q^n f(n). \tag{1.164}$$

On the other hand, one can show that if M has rank r, then the number of choices for N so that $QM = MN$ is $f(n-r)q^{r(n-r)}$. Using Exercise 1.192(b) and induction we get

$$\begin{aligned}
\#\mathcal{A} &= f(n) + \sum_{r=1}^{n} (q^n - 1) \cdots (q^n - q^{r-1}) q^{(n-r)(n-r-1)} \cdot q^{r(n-r)} \\
&= f(n) + q^{n(n-1)}(q^n - 1).
\end{aligned}$$

Comparing with equation (1.164) completes the proof.

Third solution (sketch), due to Greta Panova and Yi Sun (independently), November, 2007. Count in two ways the number of $(n+1)$-tuples $(N, v_1, v_2, \ldots, v_n)$ with N nilpotent in $\mathrm{Mat}(n, q)$, and $v_i \in \mathbb{F}_q^n$ such that $N(v_i) = v_{i+1}$ $(1 \leq i \leq n-1)$ and $v_1 \neq 0$. On the one hand there are $f(n)(q^n - 1)$ such $(n+1)$-tuples since they are determined by N and v_1. On the other hand, one can show that the number of such $(n+1)$-tuples such that $v_k \neq 0$ and $v_{k+1} = 0$ (with $v_{n+1} = 0$ always) is $f(n-k)q^{k(n-k)}(q^n - 1) \cdots (q^n - q^{k-1})$, yielding the recurrence

$$f(n)(q^n - 1) = \sum_{k=1}^{n} f(n-k) q^{k(n-k)} (q^n - 1) \cdots (q^n - q^{k-1}).$$

The proof follows straightforwardly by induction on n.

For some additional work on counting nilpotent matrices, see G. Lusztig, *Bull. London Math. Soc.* **8** (1976), 77–80.

189. See Proposition 4.27 of J. Yin, A q-analogue of Spanning Trees: Nilpotent Transformations over Finite Fields, Ph.D. thesis, M.I.T., 2009. This result may be regarded as a q-analogue of the fact that the number of spanning trees of the complete bipartite graph K_{mn} is $m^{n-1}n^{m-1}$ (see Exercise 5.30).

190. (a) We can imitate the proof of Proposition 1.10.2, using $\mathcal{I}^* := \mathcal{I} - \{x\}$ instead of $\mathcal{I}$ and β^* (defined by equation (1.112)) instead of β. We therefore get

$$\begin{aligned} \sum_{n\geq 0} \omega^*(n,q)x^n &= \prod_{n\geq 1}\prod_{j\geq 1}(1-x^{jn})^{-\beta^*(n)} \\ &= \prod_{j\geq 1}\frac{1-x^j}{1-qx^j}, \end{aligned} \tag{1.165}$$

from which the proof is immediate.

(b) This result follows easily from the Pentagonal Number Formula (1.88) and Exercise 1.74. A more careful analysis shows that if $m = \lfloor (n-1)/2 \rfloor$, then

$$\omega^*(n,q) = q^n - q^m - q^{m-1} - q^{m-2} - \cdots - q^{2\lfloor (n+5)/6 \rfloor} + O(q^{\lfloor (n+5)/6 \rfloor - 1}).$$

(c) It follows from the Pentagonal Number Formula and equation (1.165) that

$$\omega^*(n,0) = \begin{cases} (-1)^k, & \text{if } n = k(3k\pm 1)/2 \\ 0, & \text{otherwise,} \end{cases}$$

We also have

$$\omega^*(n,-1) = \begin{cases} 2(-1)^k, & \text{if } n = k^2 \\ 0, & \text{otherwise,} \end{cases}$$

a consequence of the identity (1.131) due to Gauss.

By differentiating (1.165) with respect to q and setting $q=0$, it is not hard to see that $\omega^*(n,q)$ is divisible by q^2 if and only if

$$\frac{k(3k-1)}{2} < n < \frac{k(3k+1)}{2}$$

for some $k \geq 1$.

191. *First solution* (in collaboration with G. Lusztig). Let F be an algebraic closure of $\mathbb{F}_q$. We claim that the set Ω of orbits of the adjoint representation of $\mathrm{GL}(n,F)$ has the structure

$$\Omega \cong \bigoplus_{\lambda\vdash n} F^{\ell(\lambda)}. \tag{1.166}$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, where $\lambda_k > 0$. Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in F^k$, let $M = M(\lambda, \boldsymbol{\alpha}) \in \mathrm{Mat}(n, F)$ be defined as follows: M is a direct sum of k Jordan blocks $J_1, \ldots, J_k$, with J_i containing λ_i main diagonal elements equal to α_i. We do yet have a set of orbit representatives, since if we have j blocks of the same size, then they can appear in any order. Hence the different conjugacy classes formed by j blocks of size m has the structure $F^j/\mathfrak{S}_j$, where $\mathfrak{S}_j$ acts on F^j by permuting coordinates. But it is well-known that $F^j/\mathfrak{S}_j \cong F^j$, viz., the elements of $F^j/\mathfrak{S}_j$ correspond to k-element multisets $\{\alpha_1, \ldots, \alpha_k\}$ of elements of F which we associate with $(\beta_1, \ldots, \beta_k) \in F^k$ by

$$\prod_{i=1}^{k}(x - \alpha_i) = x^k + \sum_{j=1}^{k} \beta_j x^{k-j}.$$

Hence (1.166) follows. It is now a consequence of standard properties of the Frobenius map $\alpha \mapsto \alpha^q$ that the space Ω_q of orbits of the adjoint representation of $\mathrm{GL}(n, q)$ has an analogous decomposition

$$\Omega_q \cong \bigoplus_{\lambda \vdash n} \mathbb{F}_q^{\ell(\lambda)},$$

and the proof follows.

Second solution. Let $f(z) \in \mathbb{F}_q[z]$ be a monic polynomial of degree k. Let $f(z) = \prod f_i(z)^{r_i}$ be its factorization into irreducible factors (over $\mathbb{F}_q$). Let $M_f \in \mathrm{Mat}(n, q)$ be a matrix whose adjoint orbit is indexed by $\Phi : \mathcal{I}(q) \to \mathrm{Par}$ satisfying $\Phi(f_i) = (r_i)$ (the partition with one part equal to r_i). A specific example of such a matrix is the *companion matrix*

$$M_f = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\beta_0 \\ 1 & 0 & \cdots & 0 & -\beta_1 \\ 0 & 1 & \cdots & 0 & -\beta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\beta_{k-1} \end{bmatrix},$$

where $f(z) = \beta_0 + \beta_1 z + \cdots + \beta_{k-1} z^{k-1} + z^k$. For fixed k, the space of all such M_f is just an affine space $\mathbb{F}_q^k$ (since it is isomorphic to the space of all monic polynomials of degree k). Now given a partition $\lambda \vdash n$ with conjugate $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, choose polynomials $f_i(z) \in \mathbb{F}_q[z]$ such that $\deg f_i = \lambda'_i$ and $f_{i+1} | f_i$ for all $i \geq 1$. Let $M = M_{f_1} \oplus M_{f_2} \oplus \cdots \in \mathrm{Mat}(n, q)$. For fixed λ, the space of all such M has the structure $\mathbb{F}_q^{\lambda'_1} = \mathbb{F}_q^{\ell(\lambda)}$ (since once f_{i+1} is chosen, there are $q^{\lambda'_{i+1} - \lambda'_i}$ choices for f_i). It is easy to check that the M's form a cross-section of the orbits as λ ranges over all partitions of n, so the number of orbits is $\sum_{\lambda \vdash n} q^{\ell(\lambda)}$. This argument appears in J. Hua, *J. Combinatorial Theory, Ser. A* **79** (1997), 105–117 (Theorem 11).

Third solution, due to Gabriel Tavares Bujokas and Yufei Zhao (independently), November 2007. We want the number of functions $\Phi : \mathcal{I}(q) \to \mathrm{Par}$ satisfying $\sum_{f \in \mathcal{I}(q)} |\Phi_M(f)| \cdot \deg(f) = n$. For each $i \geq 1$, let

$$p_i = \prod_{f \in \mathcal{I}(q)} f^{m_i(\Phi(f))},$$

where $m_i(\Phi(f))$ denotes the number of parts of $\Phi(f)$ equal to i. Thus the p_i's are arbitrary monic polynomials satisfying $\sum i\deg(p_i) = n$. First choose $\lambda \vdash \langle 1^{d_1}, 2^{d_2}, \ldots\rangle \vdash n$ and then each p_i so that $\deg p_i = d_i$. There are thus $q^{\sum d_i} = q^{\ell(\lambda)}$ choices for the p_i's, so

$$\omega(n,q) = \sum_{\lambda\vdash n} q^{\ell(\lambda)} = \sum_j p_j(n) q^j.$$

193. A matrix P is a projection if and only if $\Phi_P(z) = \langle 1^k\rangle$ for some k, $\Phi_P(z-1) = \langle 1^{n-k}\rangle$, and otherwise $\Phi_P(f) = \emptyset$. The proof now follows from Theorem 1.10.4 and Lemma 1.10.5 exactly as does Corollary 1.10.6.

194. A matrix M is regular if and only if for all $f \in \mathcal{I}(q)$ there is an integer $k \geq 0$ such that $\Phi_M(f) = (k)$. Write $c_f(k)$ for $c_f(\lambda)$ when $\lambda = (k)$. From Theorem 1.10.7 we have

$$c_f(k) = q^{kd} - q^{(k-1)d},\ k \geq 1,$$

where $d = \deg(f)$. Substitute $t_{f,\lambda} = 1$ if $\lambda = (k)$ and $t_{f,\lambda} = 0$ otherwise in Theorem 1.10.4 to get

$$\begin{aligned}\sum_{n\geq 0} r_n \frac{x^n}{\gamma_n} &= \prod_{f\in\mathcal{I}} \left(1 + \sum_{k\geq 1} \frac{x^{k\cdot\deg(f)}}{q^{k\cdot\deg(f)} - q^{(k-1)\cdot\deg(f)}}\right) \\ &= \prod_{d\geq 1}\left(1 + \sum_{k\geq 1}\frac{x^{kd}}{q^{kd}(1-q^{-d})}\right)^{\beta(d)} \\ &= \prod_{d\geq 1}\left(1 + \frac{(x/q)^d}{(1-q^{-d})(1-(x/q)^d)}\right)^{\beta(d)} \\ &= \prod_{d\geq 1}\left(1 + \frac{x^d}{(q^d-1)(1-(x/q)^d)}\right)^{\beta(d)}.\end{aligned}$$

We can write this identity in the alternative form

$$\sum_{n\geq 0} r_n \frac{x^n}{\gamma_n} = \frac{1}{1-x}\prod_{d\geq 1}\left(1 + \frac{x^d}{q^d(q^d-1)}\right)^{\beta(d)}$$

by using equation (1.145) with x/q substituted for x.

195. A matrix M is semisimple if and only if for all $f \in \mathcal{I}(q)$ there is an integer $k \geq 0$ such that $\Phi_M(f) = \langle 1^k\rangle$. The proof now follows from Theorem 1.10.4 and Lemma 1.10.5 exactly as does Corollary 1.10.6.

196. (a) The proof parallels that of Proposition 1.10.15. We partition $\mathfrak{S}_n$ into two classes $\mathcal{A}$ and $\mathcal{B}$, where

$$\begin{aligned}\mathcal{A} &= \{w \in \mathfrak{S}_n : w \neq 12\cdots ku \text{ for some } u \in \mathfrak{S}_{[k+1,n]}\} \\ \mathcal{B} &= \{w \in \mathfrak{S}_n : w = 12\cdots ku \text{ for some } u \in \mathfrak{S}_{[k+1,n]}\}.\end{aligned}$$

Let

$$G(n,k,q) = \{A \in \mathrm{GL}(n,q) \,:\, A_{11} + \cdots + A_{kk} = 0\}.$$

For $w \in \mathcal{A}$ we have

$$\#\left(\Gamma_w \cap G(n,k,q)\right) = \frac{1}{q}\#\Gamma_w.$$

For $w \in \mathcal{B}$ we have

$$\begin{aligned}\#\left(\Gamma_w \cap G(n,k,q)\right) &= q^{\binom{n}{2}+\mathrm{inv}(w)}(q-1)^{n-k}a_k \\ &= q^{\binom{n}{2}+\mathrm{inv}(w)}(q-1)^{n-k}((q-1)^k + (-1)^k(q-1)).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{w\in\mathcal{B}} \#(\Gamma_w \cap G(n,k,q)) &= \frac{1}{q}\left(\sum_{w\in\mathcal{B}} q^{\binom{n}{2}+\mathrm{inv}(w)}(q-1)^n \right.\\ &\left. + (-1)^k(q-1)q^{\binom{n}{2}-\binom{n-k}{2}} \sum_{u\in\mathfrak{S}_{[k+1,n]}} q^{\binom{n-k}{2}+\mathrm{inv}(u)}(q-1)^{n-k}\right) \\ &= \frac{1}{q}\left(\sum_{w\in\mathcal{B}}(\#\Gamma_w) + (-1)^k(q-1)q^{\frac{1}{2}k(2n-k-1)}\gamma_{n-k}(q)\right),\end{aligned}$$

and the proof follows.

(b) The hyperplane H can be defined by $H = \{M \in \mathrm{Mat}(n,q) \,:\, M\cdot N = 0\}$, where N is a fixed nonzero matrix in $\mathrm{Mat}(n,q)$ and $M \cdot N = \mathrm{tr}(MN^t)$, the standard dot product in the vector space $\mathrm{Mat}(n,q)$. If $P, Q \in \mathrm{GL}(n,q)$, then $M \cdot N = 0$ if and only if $((P^t)^{-1}M(Q^t)^{-1}) \cdot (PNQ) = 0$. Since two matrices $N, N' \in \mathrm{Mat}(n,q)$ are related by $N' = PNQ$ for some $P, Q \in \mathrm{GL}(n,q)$ if and only if they have the same rank, it follows that $\#(\mathrm{GL}(n,q) \cap H)$ depends only on $\mathrm{rank}(N)$. If $\mathrm{rank}(N) = k$, then we may take

$$N_{ij} = \begin{cases} 1, & 1 \le i = j \le k \\ 0, & \text{otherwise.}\end{cases}$$

Hence $\#(\mathrm{GL}(n,q) \cap H)$ is given by the right-hand side of equation (1.150).

197. *Hint.* Let $f(n,k)$ be the number of $k \times n$ matrices over $\mathbb{F}_q$ of rank k with zero diagonal, where $1 \le k \le n$. Show that

$$f(n,k+1) = q^{k-1}(q-1)(f(n,k)\cdot \boldsymbol{(n-k)} - f(n-1,k)),$$

with the initial condition $f(n,1) = q^{n-1} - 1$. The solution to this recurrence is

$$f(n,k) = q^{\binom{k-1}{2}-1}(q-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\boldsymbol{(n-i)!}}{\boldsymbol{(n-k)!}}\right).$$

Now set $k = n$.

This result is due to J. B. Lewis, R. I. Liu, A. H. Morales, G. Panova, S. V. Sam, and Y. Zhang, Matrices with restricted entries and q-analogues of permutations, `arXiv:1011.4539`. (Proposition 2.2). This paper contains a host of other results about counting matrices over $\mathbb{F}_q$. A further result in this paper is given by Exercise 1.199.

198. (a) An $(n+1)\times(n+1)$ symmetric matrix may be written as

$$N = \begin{bmatrix} \beta & y \\ y^t & M \end{bmatrix},$$

where M is an $n\times n$ symmetric matrix, $\beta\in\mathbb{F}_q$, and $y\in\mathbb{F}_q^n$. Elementary linear algebra arguments show that from a particular matrix M of rank r we obtain:

- $q^{n+1}-q^{r+1}$ matrices N of rank $r+2$,
- $(q-1)q^r$ matrices N of rank $r+1$,
- q^r matrices N of rank r,
- no matrices of other ranks.

The recurrence (1.151) follows. This recurrence (with more details of the proof) was given by J. MacWilliams, *Amer. Math. Monthly* **76** (1969), 152–164, and was used to prove (b). A simpler recurrence for $h(n,n)$ alone was given by G. Lusztig, *Transformation Groups* **10** (2005), 449–487 (end of §3.14).

(b) We can simply verify that the stated formula for $h(n,r)$ satisfies the recurrence (1.151), together with the initial conditions. For some generalizations and further information, see R. Stanley, *Ann. Comb.* **2** (1998), 351–363; J. R. Stembridge, *Ann. Comb.* **2** (1998), 365–385; F. Chung and C. Yang, *Ann. Comb.* **4** (2000), 13–25; and P. Belkale and P. Brosnan, *Duke Math. J.* **116** (2003), 147–188.

NOTE. There is less *ad hoc* way to compute the quantity $h(n,n)$. Namely, $\mathrm{GL}(n,q)$ acts on $n\times n$ invertible symmetric matrices M over $\mathbb{F}_q$ by $A\cdot M = A^tMA$. This action has two orbits whose stabilizers are the two forms of the orthogonal group $O(n,q)$. The orbit sizes can be easily computed from standard facts about $O(n,q)$. For further details, see R. Stanley, *op. cit.* (§4).

199. (a) The equality of the first two items when q is even is due to J. MacWilliams, *Amer. Math. Monthly* **76** (1969), 152–164 (Theorems 2, 3). The equality of the second two items appears in O. Jones, *Pacific J. Math.* **180** (1997), 89–100. For the remainder of the exercise, see Section 3 of the paper of Lewis-Liu-Morales-Panova-Sam-Zhang cited in the solution to Exercise 1.197.

200. This result was conjectured by A. A. Kirillov and A. Melnikov, in *Algèbre non commutative, groupes quantiques et invariants (Reims, 1995)*, Sémin. Congr. **2**, Soc. Math. France, Paris, 1997, pp. 35–42, and proved by S. B. Ekhad and D. Zeilberger, *Electronic J. Combinatorics* **3**(1) (1996), R2. No conceptual reason is known for such a simple formula.

201. (a) The result follows from the theory of Gauss sums as developed e.g. in K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed.,

Springer-Verlag, New York, 1990, and may have been known to Gauss or Eisenstein. This information was provided by N. Elkies (private communication, 1 August 2006).

(b) The argument is analogous to the proof of Proposition 1.10.15. Let

$$G = \{A \in \mathrm{GL}(3,q) \,:\, \mathrm{tr}(A) = 0,\ \det(A) = 1\}.$$

If $123 \neq w \in \mathfrak{S}_3$, then $\#(\Gamma_2 \cap G) = \frac{1}{q(q-1)}\#\Gamma_w$. On the other hand, $\#(\Gamma_{123} \cap G) = q^3 f(q)$. Hence we get

$$\begin{aligned} \#G &= \frac{1}{q(q-1)}(\gamma(3,q) - \#\Gamma_{123}) + q^3 f(q) \\ &= q^3(q-1)^2(q^2+2q+2) + q^3 f(q). \end{aligned}$$

202. This result is an instance of the Shimura-Taniyama-Weil conjecture, viz., every elliptic curve is modular. An important special case of the conjecture (sufficient to imply Fermat's Last Theorem) was proved by A. Wiles in 1993, with a gap fixed by Wiles and R. Taylor in 1994. The full conjecture was proved by Breuil, Conrad, Diamond, and Taylor in 1999. Our example follows H. Darmon, *Notices Amer. Math. Soc.* **46** (1999), 1397–1401, which has much additional information.

203. The statement about 103,049 was resolved in January, 1994, when David Hough, then a graduate student at George Washington University, noticed that 103,049 is the total number of bracketings of a string of 10 letters. The problem of finding the number of bracketing of a string of n letters is known as *Schröder's second problem* and is discussed in Section 6.2. See also the Notes to Chapter 6, where also a possible interpretation of 310,952 is discussed. Hough's discovery was first published by R. Stanley, *Amer. Math. Monthly* **104** (1997), 344–350. A more scholarly account was given by F. Acerbi, *Archive History Exact Sci.* **57** (2003), 465–502.