姓名: Solution

葉均承　應數一線性代數

學號: ______

Quiz 2

考試日期: 2020/03/26

不可使用手機、計算器，禁止作弊!
背面還有題目

1. (50%) (a) Solve the system $\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 4x_1 - 2x_2 \end{cases}$ $\quad \vec{X}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \overset{A}{\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{X}$

   (b) Find the solution that satisfies the initial condition $x_1(0) = 1, x_2(0) = 6$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

$$\Rightarrow \lambda = 2, -3$$

$\boxed{\lambda = 2}$

$$A - 2I = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

let $x_2 = r$, $-x_1 + x_2 = 0$

$\Rightarrow x_1 = r$

$\therefore$ eigenvector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{\lambda = -3}$

$$A + 3I = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}$$

let $x_2 = r$, $4x_1 + x_2 = 0 \Rightarrow x_1 = -\frac{1}{4}r$

$\therefore$ eigenvector $\vec{v} = \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$$\therefore C = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \text{ and } A = CDC^{-1}$$

$$\because \vec{X}' = A\vec{X} = CDC^{-1}\vec{X} \Rightarrow \underbrace{(C^{-1}\vec{X}')}_{\vec{y}'} = D\underbrace{(C^{-1}\vec{X})}_{\vec{y}} \quad \therefore \vec{y}' = D\vec{y}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{cases} y_1' = 2y_1 \\ y_2' = -3y_2 \end{cases} \Rightarrow \begin{cases} y_1 = k_1 e^{2t} \\ y_2 = k_2 e^{-3t} \end{cases} \quad \therefore \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k_1 e^{2t} \\ k_2 e^{-3t} \end{bmatrix}$$

$$\because \vec{y} = C^{-1}\vec{X} \quad \therefore \vec{X} = C\vec{y} = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} k_1 e^{2t} \\ k_2 e^{-3t} \end{bmatrix} = \begin{bmatrix} k_1 e^{2t} - k_2 e^{-3t} \\ k_1 e^{2t} + 4k_2 e^{-3t} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} 1 = x_1(0) = k_1 - k_2 \\ 6 = x_2(0) = k_1 - 4k_2 \end{cases} \Rightarrow 5k_2 = 5 \Rightarrow k_2 = 1 \Rightarrow k_1 = 2$$

$$\therefore \vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2e^{2t} - e^{-3t} \\ 2e^{2t} + 4e^{-3t} \end{bmatrix}$$

驗算

① $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4e^{2t} + 3e^{-3t} \\ 4e^{2t} - 12e^{-3t} \end{bmatrix}$ ② $\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 4x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} (2e^{2t} - e^{-3t}) + (2e^{2t} + 4e^{-3t}) \\ 4(2e^{2t} - e^{-3t}) - 2(2e^{2t} + 4e^{-3t}) \end{bmatrix}$ ①=② ok

2. (50%)Let

$$A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$$

Find $A^5 = \begin{bmatrix} -23 & 55 \\ 220 & -188 \end{bmatrix}$

by (1) $C = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$, $A = CDC^{-1}$

Note : $\det(C) = 5$

$\Rightarrow C^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}$

$\therefore A^5 = CD^5C^{-1}$

$= \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 32 & 0 \\ 0 & -243 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 32 & 243 \\ 32 & -972 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} -115 & 275 \\ 1100 & -940 \end{bmatrix}$

$= \begin{bmatrix} -23 & 55 \\ 220 & -188 \end{bmatrix}$