

應數一線性代數 2020 秋, 期末考 解答

學號: \_\_\_\_\_, 姓名: \_\_\_\_\_

本次考試共有 8 頁 (包含封面), 有 11 題。如有缺頁或漏題, 請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號, 並在每一頁的最上方屬名, 避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程, 閱卷人員會視情況給予部份分數。  
沒有計算過程, 就算回答正確答案也不會得到滿分。  
答卷請清楚乾淨, 儘可能標記或是框出最終答案。

高師大校訓: 誠敬宏遠

誠, 一生動念都是誠實端正的。      敬, 就是對知識的認真尊重。  
宏, 開拓視界, 恢宏心胸。          遠, 任重致遠, 不畏艱難。

請尊重自己也尊重其他同學, 考試時請勿東張西望交頭接耳。

1. (10 points) Find the determinant of

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$

Answer: -30

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 3 \end{vmatrix} &= 1 \times \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 3 \end{vmatrix} \quad (\text{expand on the } 3^{\text{rd}} \text{ column.}) \\ &= 1 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \times \begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} \quad (\text{by 4-3 problem 11}) \\ &= 1 \times (-2 - 3) \times (6 - 0) = -30 \end{aligned}$$

2. (10 points) Suppose that A is a  $5 \times 5$  matrix with determinant 7.

(a) Find  $\det(3A) = \underline{3^5 \times 7 = 1701}$

(b) Find  $\det(A^{-1}) = \underline{1/7}$

(c) Find  $\det(2A^{-1}) = \underline{2^5 \times 1/7 = 32/7}$

(d) Find  $\det((2A)^{-1}) = \underline{1/(2^5 \times 7) = 1/224}$

3. (5 points) Suppose that  $A$  is a  $3 \times 3$  matrix with row vectors  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$ , and that  $\det(A) = 3$ . Find the determinant of the matrix having  $\vec{a}$ ,  $\vec{b}$ ,  $2\vec{a} + 3\vec{b} + 2\vec{c}$  as its row vectors

Determinant = 6

$$\begin{aligned}
 \begin{vmatrix} \vec{a} \\ \vec{b} \\ 2\vec{a} + 3\vec{b} + 2\vec{c} \end{vmatrix} &= \begin{vmatrix} \vec{a} \\ \vec{b} \\ 3\vec{b} + 2\vec{c} \end{vmatrix} \quad (R_3 = R_3 - 2 \times R_1) \\
 &= \begin{vmatrix} \vec{a} \\ \vec{b} \\ 2\vec{c} \end{vmatrix} \quad (R_3 = R_3 - 3 \times R_2) \\
 &= 2 \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix} \\
 &= 2 \times 3 = 6
 \end{aligned}$$

4. (10 points)

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

The inverse of  $A = \underline{\begin{bmatrix} 0.5 & -0.5 & 0 \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}}$ , and the adjoint matrix of  $A = \underline{\begin{bmatrix} 1 & -1 & 0 \\ -2 & 6 & -2 \\ 2 & -4 & 2 \end{bmatrix}}$

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

$$\text{adj}(A) = [a'_{i,j}]^T, \text{ where } a'_{i,j} = (-1)^{i+j} \det(A_{i,j})$$

5. (5 points) Let  $\vec{a} = \vec{i} - 3\vec{k}$ ,  $\vec{b} = -\vec{i} + 4\vec{j}$ ,  $\vec{c} = \vec{i} + \vec{j} + \vec{k}$ .

Find  $\vec{a} \cdot (\vec{b} \times \vec{c}) =$  19

$$\vec{i} = [1, 0, 0], \vec{j} = [0, 1, 0], \vec{k} = [0, 0, 1],$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix} = [4, 1, -5]$$

$$[1, 0, -3] \cdot [4, 1, -5] = 19$$

6. (10 points) Find out whether points  $(1, 2, 1)$ ,  $(3, 3, 4)$ ,  $(2, 2, 2)$  and  $(4, 3, 5)$  lie in a plane in  $\mathbb{R}^3$

Answer: Yes

Name  $A(1, 2, 1)$ ,  $B(3, 3, 4)$ ,  $C(2, 2, 2)$  and  $D(4, 3, 5)$ ,  
then  $\vec{AB} = [2, 1, 4]$ ,  $\vec{AC} = [1, 0, 1]$ ,  $\vec{AD} = [3, 1, 4]$

$$\begin{vmatrix} \vec{AB} \\ \vec{AC} \\ \vec{AD} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 0$$

7. (10 points) Using **Cramer's rule** to find the component  $y$  of the solution vector for the given linear system.

$$\begin{cases} 2x - 3y = 1 \\ -4x + 6y = -2 \end{cases}$$

$$y = \underline{\text{無限多組解 or } \frac{2x-1}{3}}$$

$$|A| = \begin{vmatrix} 2 & -3 \\ -4 & 6 \end{vmatrix} = 0, |B_2| = \begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} = 0,$$

8. (10 points) Circle True or False. Read each statement in original Greek before answering.

- (a) True False There's an unique coordinate vector associated with each vector  $\vec{v} \in V$  relative to a basis for  $V$
- (b) True False A linear transformation  $T : V \rightarrow V'$  carries the zero vector of  $V$  into the zero vector of  $V'$ .
- (c) True False The parallelogram (平行四邊形) in  $\mathbb{R}^2$  determined by non-zero vectors  $\vec{a}, \vec{b}$  is a square (正方形) if and only if  $\vec{a} \cdot \vec{b} = 0$
- (d) True False The product of a square matrix and its adjoint is the identity matrix.
- (e) True False There is no square matrix  $A$  such that  $\det(A^T A) = -1$ .

(a) It needs to be an **ordered basis** to be TRUE.

(c)  $\vec{a} = [1, 0], \vec{b} = [0, 2] \Rightarrow \vec{a} \cdot \vec{b} = 0$ , but it's not a square.

(d) By Theorem 4.6:  $A * adj(A) = \det(A)I$

9. (10 points) Let  $V$  and  $V''$  be vector spaces with ordered bases  $B = ([1, 3, -2], [4, 1, 2], [-1, 1, 0])$  and  $B' = ([1, 0, 1, 0], [2, 1, 1, -1], [0, 1, 1, -1], [2, 0, 3, 1])$ , respectively, and let  $T : V \rightarrow V'$  be the linear transformation having the given matrix  $A$  as matrix representation relative to  $B, B'$ . Find  $T([0, 3, -6])$ .

$$A = \begin{bmatrix} 0 & 4 & -1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

(a) If  $\vec{v} = [0, 3, -6]$ , then  $\vec{v}_B = \underline{[2, -1, -2]}$ .

(b)  $T([0, 3, -6]) = \underline{[-14, -1, -12, -2]}$ .

$$\left[ \begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 3 & 1 & 1 & 3 \\ -2 & 2 & 0 & -6 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -14 \\ -1 \\ -12 \\ -2 \end{bmatrix}$$

10. (10 points) Let  $T : P_3 \longrightarrow P_2$  be defined by  $T(p(x)) = D(p(x+1))$ , and let  $B = (x^3, x^2, x, 1)$  and  $B' = (x^2, x, 1)$ .

(a) Find the matrix  $A$  as matrix representation of  $T$  relative to  $B, B'$ .  $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$ .

(b) Use  $A$  to compute  $T(4x^3 - 5x^2 + 3x - 2) = \underline{12x^2 + 14x + 5}$ .

(a)  $T(x^3) = 3x^2 + 6x + 3, T(x^2) = 2x + 2, T(x^1) = 1, T(x^0) = 0$

(b)

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \\ 5 \end{bmatrix}$$

- (b) Find a basis for the subspace of  $F$  generated by the functions  $\{f_1, f_2, f_3, f_4\}$ , where

$$f_3(x) = 4 \cos(x) - 5 \sin(2x) + 3 \cos(2x), \quad f_4(x) = 1 + 2 \sin(2x)$$

- $$x = \frac{3\pi}{2}, \quad a - b + c + 0 - e = 0 \quad (4)$$

Hence, we have  $a = b = c = d = e = 0$ . Therefore, its independent.

- $$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & -2 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ -1 & 0 & -5 & 2 \\ -3 & 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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[illegible]