姓名: SOLUTION　　　　　　　　　　　　　　　　　葉均承　　應數一線性代數

學號: ________________

Quiz 12

考試日期: 2021/06/02

此為開書考，但是禁止與其他人討論
請框出答案.　不可使用手機、計算器，禁止作弊!

1. Find an unitary matrix U and a diagonal matrix D such that $D = U^{-1}AU$, where

$$A = \begin{bmatrix} 2 & 0 & 1-i \\ 0 & 3 & 0 \\ 1+i & 0 & 1 \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1-i \\ 0 & 3-\lambda & 0 \\ 1+i & 0 & 1-\lambda \end{vmatrix} = -\lambda(3-\lambda)^2$$

$\boxed{\lambda = 0}$

$$rref(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2}-\frac{1}{2}i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} -\frac{1}{2}+\frac{1}{2}i \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{\lambda = 3}$

$$rref(A-3I) = \begin{bmatrix} 1 & 0 & -1+i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v} = r\begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \lambda_1 = 0,\ \vec{v}_1 = \begin{bmatrix} -\frac{1}{2}+\frac{1}{2}i \\ 0 \\ 1 \end{bmatrix},\ \lambda_2 = \lambda_3 = 3,\ \vec{v}_2 = \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix},\ \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Notice that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthogonal.

$$\therefore \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix} -\frac{1}{2}+\frac{1}{2}i \\ 0 \\ 1 \end{bmatrix},\ \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix},\ \vec{u}_3 = \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$U = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} -\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i & 1-i & 0 \\ 0 & 0 & \sqrt{3} \\ \sqrt{2} & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

2. Prove the following two statement.

   (a) The product of two commuting $n \times n$ Hermitian matrices is also a Hermitian matrix.

   (b) The product of two $n \times n$ unitary matrices is also a unitary matrix.

(a) Let H_1, H_2 are Hermitian matrices, i.e. $H_1^* = H_1, H_2^* = H_2$. Since $H_1 1, H_2$ are commuting, i.e. $H_1 H_2 = H_2 H_1$ $(H_1 H_2)^* = H_2^* H_1^* = H_2 H_1 = H_1 H_2$. Hence $H_1 H_2$ is a Hermitian matrix.

(b) Let U_1, U_2 are unitary matrices, i.e. $U_1^* U_1 = I, U_2^* U_2 = I$. $(U_1 U_2)^*(U_1 U_2) = U_2^* U_1^* (U_1 U_2) = U_2^* I U_2$. Hence $U_1 U_2$ is a unitary matrix.

由於遠距教學，此為線上開書考試，請在答題後，拍照上傳到以下網址：
https://forms.gle/VrrVUnh5p1kbhYPC8