姓名: SOLUTION 葉均承 應數一線性代數

學號: ____________

Quiz 13

考試日期: 2021/12/30

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!
3. 作答完畢請拍照上傳 Googld Classroom
4. 照片請清晰並轉正

1. Let A be a 3×3 matrix with row vectors $\vec{a}, \vec{b}, \vec{c}$ and with determinant equal to 5. Find the determinant of the following matrices.

   (a) B is the matrix having row vectors $\vec{a}, 3\vec{a} + 4\vec{b} + 2\vec{c}, \vec{b}$.

   (b) C is the matrix having row vectors $\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{a} + \vec{c}$.

   Answer: = $\det(B) = -10, \det(C) = 10$.

   Let's present the matrices graphically.

$$A = \begin{bmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{b} & - \end{bmatrix}, B = \begin{bmatrix} - & \vec{a} & - \\ - & 3\vec{a} + 4\vec{b} + 2\vec{c} & - \\ - & \vec{b} & - \end{bmatrix}, C = \begin{bmatrix} - & \vec{a} + \vec{b} & - \\ - & \vec{b} + \vec{c} & - \\ - & \vec{c} + \vec{a} & - \end{bmatrix}$$

   By the Property 1, Property 2, Property 3, Property 4 and Property 5 in Section 4-3

$$\begin{aligned}
\det(B) &= \det(\begin{bmatrix} - & \vec{a} & - \\ - & 3\vec{a} + 4\vec{b} + 2\vec{c} & - \\ - & \vec{b} & - \end{bmatrix}) \\
&= \det(\begin{bmatrix} - & \vec{a} & - \\ - & 4\vec{b} + 2\vec{c} & - \\ - & \vec{b} & - \end{bmatrix}) && (R_2 \to R_2 - 3R_1) \\
&= \det(\begin{bmatrix} - & \vec{a} & - \\ - & 2\vec{c} & - \\ - & \vec{b} & - \end{bmatrix}) && (R_2 \to R_2 - 4R_3) \\
&= 2\det(\begin{bmatrix} - & \vec{a} & - \\ - & \vec{c} & - \\ - & \vec{b} & - \end{bmatrix}) && (R_2 \to \frac{1}{2}R_2) \\
&= -2\det(\begin{bmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{bmatrix}) && (R_2 \leftrightarrow R_) \\
&= -2\det(A) = -2 * 5 = -10
\end{aligned}$$

$$
\begin{aligned}
\det(C) &= \det(\begin{bmatrix} - & \vec{a}+\vec{b} & - \\ - & \vec{b}+\vec{c} & - \\ - & \vec{c}+\vec{a} & - \end{bmatrix}) \\
&= \det(\begin{bmatrix} - & 2(\vec{a}+\vec{b}+\vec{c}) & - \\ - & \vec{b}+\vec{c} & - \\ - & \vec{c}+\vec{a} & - \end{bmatrix}) && (R_1 \to R_1 + R_2), (R_1 \to R_1 + R_3) \\
&= 2\det(\begin{bmatrix} - & \vec{a}+\vec{b}+\vec{c} & - \\ - & \vec{b}+\vec{c} & - \\ - & \vec{c}+\vec{a} & - \end{bmatrix}) && (R_1 \to \frac{1}{2}R_1) \\
&= 2\det(\begin{bmatrix} - & \vec{a}+\vec{b}+\vec{c} & - \\ - & -\vec{a} & - \\ - & -\vec{b} & - \end{bmatrix}) && (R_2 \to R_2 - R_1), (R_3 \to R_3 - R_1) \\
&= 2\det(\begin{bmatrix} - & \vec{c} & - \\ - & -\vec{a} & - \\ - & -\vec{b} & - \end{bmatrix}) && (R_1 \to R_1 - R_2), (R_1 \to R_1 - R_3) \\
&= 2(-1)^2\det(\begin{bmatrix} - & \vec{c} & - \\ - & \vec{a} & - \\ - & \vec{b} & - \end{bmatrix}) && (R_2 \to -R_2), (R_3 \to -R_3) \\
&= 2\det(\begin{bmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{bmatrix}) && (R_1 \leftrightarrow R_2), (R_2 \leftrightarrow R_3) \\
&= 2\det(A) = 2 * 5 = 10
\end{aligned}
$$

2. Prove that if A is invertible, then $\det(A) = 1/\det(A)$.

Since A is invertible, then A^{-1} exists and by Theorem 4.3 in Section 4-3:

A square matrix A is invertible if and only if $\det(A) \neq 0$.

By Theorem 4.4 in Section 4-3 :

For two matrix $A, B, \det(AB) = \det(A)\det(B)$

, we have $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$.
Hence $\det(A^{-1}) = 1/\det(A)$.