學號: ______________

Quiz 13

考試日期: 2022/06/09

不可使用手機、計算器，禁止作弊!

1. Use the process in Schur's Lemma to find an unitary matrix U such that $U^{-1}AU$ is an upper triangular.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

(a) Since the first column of A is $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, we can consider the first step of the Shur's lemma is done!

Pick $U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Let

$$A = \left[\begin{array}{c|cc} 2 & 1 & -2 \\ \hline 0 & 1 & 3 \\ 0 & 1 & -1 \end{array}\right] = \left[\begin{array}{c|cc} 2 & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\tilde{A}$}} \\ 0 & & \end{array}\right],\ \tilde{A} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$$

$$|\tilde{A} - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)$$

$$\tilde{A} + 2I \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ \vec{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Pick $\vec{q}_2$ such that $\vec{q}_2$ is perpendicular to $\vec{q}_1$. Pick $\vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\tilde{U} = [\vec{q}_1 \quad \vec{q}_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},\ \text{and } \tilde{U}^* \tilde{A} \tilde{U} = \begin{bmatrix} -2 & * \\ 0 & * \end{bmatrix}$$

$$U_2 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\tilde{U}$}} \\ 0 & & \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(c) Combine (a) and (b).

$$U = U_1 U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Check:

$$U^* A U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^* \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & -2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

特別注意一下，這題的 U 跟乘完後的上三角矩陣都不是唯一的。