不可使用手機、計算器，禁止作弊!

1. Let A be a 3×3 matrix with row vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and with determinant equal to 8. Find the determinant of the following matrices.
   (a) Let B is the matrix having row vectors $\vec{a}+5\vec{b}, 3\vec{a}+5\vec{b}-2\vec{c}, \vec{b}$. $det(B)$= **16** .
   (b) Let C is the matrix having row vectors $\vec{a}+\vec{a}$, $\vec{b}+\vec{c}$, $\vec{a}+\vec{c}$. $\det(C)$= **16** .
   (c) Let D is A^{-1}. $\det(D)$= **1/8** .
   (d) Let E is A^T. $\det(EA)$= 64 .
   (e) Let F is $5A$. $\det(F)$= $5^3\times 8 = 1000$.

Solution：

Let's present the matrices graphically.

$$A=\begin{bmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{b} & - \end{bmatrix}, B=\begin{bmatrix} - & \vec{a}+5\vec{b} & - \\ - & 3\vec{a}+5\vec{b}-2\vec{c} & - \\ - & \vec{b} & - \end{bmatrix}, C=\begin{bmatrix} - & \vec{a}+\vec{a} & - \\ - & \vec{b}+\vec{c} & - \\ - & \vec{a}+\vec{c} & - \end{bmatrix}$$

By the Property 1, Property 2, Property 3, Property 4 and Property 5 in Section 4-3

$$\begin{aligned}
\det(B) &= \det(\begin{bmatrix} - & \vec{a}+5\vec{b} & - \\ - & 3\vec{a}+5\vec{b}-2\vec{c} & - \\ - & \vec{b} & - \end{bmatrix}) \\
&= \det(\begin{bmatrix} - & \vec{a} & - \\ - & 3\vec{a}+5\vec{b}-2\vec{c} & - \\ - & \vec{b} & - \end{bmatrix}) && (R_1 \to R_1 - 5R_1) \\
&= \det(\begin{bmatrix} - & \vec{a} & - \\ - & \vec{a}-2\vec{c} & - \\ - & \vec{b} & - \end{bmatrix}) && (R_2 \to R_2 - 5R_3) \\
&= \det(\begin{bmatrix} - & \vec{a} & - \\ - & -2\vec{c} & - \\ - & \vec{b} & - \end{bmatrix}) && (R_2 \to R_2 - R_1) \\
&= -2\det(\begin{bmatrix} - & \vec{a} & - \\ - & \vec{c} & - \\ - & \vec{b} & - \end{bmatrix}) && (R_2 \to \frac{-1}{2}R_2) \\
&= 2\det(\begin{bmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{bmatrix}) && (R_2 \leftrightarrow R_3) \\
&= 2\det(A) = 2*8 = 16
\end{aligned}$$

$$\begin{aligned}
\det(C) &= \det(\begin{bmatrix} - & \vec{a}+\vec{a} & - \\ - & \vec{b}+\vec{c} & - \\ - & \vec{a}+\vec{c} & - \end{bmatrix}) \\
&= \det(\begin{bmatrix} - & 2\vec{a} & - \\ - & \vec{b}+\vec{c} & - \\ - & \vec{a}+\vec{c} & - \end{bmatrix}) \\
&= 2\det(\begin{bmatrix} - & \vec{a} & - \\ - & \vec{b}+\vec{c} & - \\ - & \vec{a}+\vec{c} & - \end{bmatrix}) && (R_1 \to \frac{1}{2}R_1) \\
&= 2\det(\begin{bmatrix} - & \vec{a} & - \\ - & \vec{b}+\vec{c} & - \\ - & \vec{c} & - \end{bmatrix}) && (R_3 \to R_3 - R_1) \\
&= 2\det(\begin{bmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{bmatrix}) && (R_2 \to R_2 - R_3) \\
&= 2\det(A) = 2 * 8 = 16
\end{aligned}$$

2. Find the value of λ for the given matrix is singular.

$$\begin{bmatrix} 2-\lambda & 3 \\ 1 & \lambda \end{bmatrix}$$

Answer: $\lambda = (1+\sqrt{2}i),\ (1-\sqrt{2}i)$ **or "no real solution"**

Solution：

Since A is invertible, then A^{-1} exists and by Theorem 4.3 in Section 4-3:

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Since the matrix is singular, we have

$$0 = \det\left(\begin{bmatrix} 2-\lambda & 3 \\ 1 & \lambda \end{bmatrix}\right) = (2-\lambda)\lambda - 3\times 1 = -\lambda^2 + 2\lambda - 3$$

It is easy to find that $\lambda = (1+\sqrt{2}i),\ (1-\sqrt{2}i)$ are the only two solution. Therefore, the given matrix is singular only if $\lambda = (1+\sqrt{2}i),\ (1-\sqrt{2}i)$.