不可使用手機、計算器，禁止作弊!

1. Let F bet he set of all real-valued functions on a (nonempty) set S; that is , let F be the set of all functions mapping S into $\mathbb{R}$. For $f, g \in F$, let the sum $f \oplus g$ of two functions f and g in F, and for any scalar r, let scalar multiplication be defined below. Is this set a vector space?

$$(f \oplus g)(x) = 2f(x) + 2g(x) \quad \text{for all } x \in S$$
$$(r \otimes f)(x) = rf(x) - r \quad \text{for all } x \in S$$

   a. Is this set a vector space? **NO!**
      Hint: Show by verifying the closed under two operations, A1-A4 and S1-S4.

   b. What is the zero vector in this vector space?
      Answer: the zero vector is _____ , for any functions f, the $-f$ is _____

Solution：

A0 $(f \oplus g)(x) = 2f(x) + 2g(x)$ and **S0** $(r \otimes f)(x) = rf(x) - r$ are both real-valued functions on $\mathbb{R}$, hence proved the closed.

Let $f(x), g(x), h(x)$ are functions in F, and r, s are scalar in $\mathbb{R}$. We can check:

A3 $\vec{0} = 0 \otimes f(x) = 0f(x) - 0 = 0.$
$\vec{0} \oplus f(x) = 2 \times 0 + 2f(x) = 2f(x) \neq f(x)$

A4 $-f(x) = (-1) \otimes f(x) = -f(x) - (-1) = 1 - f(x).$
$(f \oplus (-f))(x) = 2f(x) + 2(1 - f(x)) = 2 \neq \vec{0}$

The properties **A4** and **A3** are NOT hold. Therefore, (F, $\oplus$, $\otimes$) is NOT a vector space.

2. Consider the set $\mathbb{R}^2$, with the addition defined by $[x,y] \oplus [a,b] = [x+a+1, y+b]$, and with scalar multiplication defined by $r \otimes [x,y] = [rx+r-1, ry]$.

   a. Is this set a vector space? **Yes!**
   Hint: Show by verifying the closed under two operations, A1-A4 and S1-S4.

   b. What is the zero vector in this vector space? *Hint:* The zero vector will NOT be the vector $[0,0]$.
   Answer: the zero vector is $\vec{0} = [-1,0]$, for any vectors [x,y], the -[x,y] is $[-x-2, -y]$

Solution：

A0 $[x,y] \oplus [a,b] = [x+a+1, y+b]$ and **S0** $r \otimes [x,y] = [rx+r-1, ry]$ are both in $\mathbb{R}^2$, hence proved the closed.

A1 $([x,y] \oplus [a,b]) \oplus [p,q] = [x+a+1, y+b] \oplus [p,q] = [x+a+1+p+1, y+b+q] = [x+a+p+1+1, y+b+q] = [x,y] \oplus [a+p+1, b+q] = [x,y] \oplus ([a,b] \oplus [p,q])$

A2 $[x,y] \oplus [a,b] = [x+a+1, y+b] = [a+x+1, b+y] = [a,b] \oplus [x,y]$

A3 $\vec{0} = 0 \otimes [x,y] = [0x+0-1, 0y] = [-1,0]$.
$\vec{0} \oplus [x,y] = [-1,0] \oplus [x,y] = [-1+x+1, 0+y] = [x,y]$

A4 $-[x,y] = (-1) \otimes [x,y] = [-x+(-1)-1, -y] = [-x-2, -y]$.
$[x,y] \oplus [-x-2, -y] = [x-x-2+1, y-y] = [-1,0]$

S1 $r \otimes ([x,y] \oplus [a,b]) = r \otimes [x+a+1, y+b] = [rx+ra+r+r-1, ry+rb] = [rx+r-1+ra+r-1+1, ry+rb] = [rx+r-1, ry] \oplus [ra+r-1, rb] = (r \otimes [x,y]) \oplus (r \otimes [a,b])$

S2 $(r+s) \otimes [x,y] = [(r+s)x+(r+s)-1, (r+s)y] = [rx+sx+r+s-1, ry+sy] = [rx+r-1+sx+s-1+1, ry+sy] = (r \otimes [x,y]) \oplus (r \otimes [a,b])$

S3 $s \otimes (r \otimes [x,y]) = s \otimes ([rx+r-1, ry]) = [srx+sr-s+s-1, sry] = (rs) \otimes [x,y]$

S4 $1 \otimes [x,y] = [x+1-1, y] = [x,y]$