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Quiz 16

考試日期: 2023/06/14

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Find a Jordan canonical form and a Jordan basis for the matrix A

$$A = \begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Solution：

It is easy to find that the A has the eigenvalue i, whose algebraic multiplicity is 2 and A has the eigenvalue 2, whose algebraic multiplicity is 3.

$$A - iI = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2-i & 0 & 0 \\ 0 & 0 & 0 & 2-i & 0 \\ 2 & 0 & -1 & 0 & 2-i \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & 0 & 2-i \\ 0 & 0 & 2-i & 0 & 0 \\ 0 & 0 & 0 & 2-i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. nullity $= 2$ and $null(A - iI) = sp(\vec{e}_2, \begin{bmatrix} -2+i \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}) = sp(\vec{b}_1, \vec{b}_2)$.

From above, we know that

$$\begin{aligned} (A - iI) : & \vec{b}_1 \to \vec{0} \\ & \vec{b}_2 \to \vec{0} \end{aligned}$$

Thus, pick $\vec{b}_1 = \vec{e}_2$, and $\vec{b}_2 = \begin{bmatrix} -2+i \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$.

$$A-2I=\begin{bmatrix}-2+i & 0 & 0 & 0 & 0\\ 0 & -2+i & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 2 & 0 & -1 & 0 & 0\end{bmatrix}, \text{ nullity} = 2\ , \qquad null(A-2I)=sp(\vec{e}_4,\vec{e}_5)$$

$$(A-2I)^2=\begin{bmatrix}3-4i & 0 & 0 & 0 & 0\\ 0 & 3-4i & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -4+2i & 0 & 0 & 0 & 0\end{bmatrix}, \text{ nullity} = 3\ , \quad null((A-2I)^2)=sp(\vec{e}_4,\vec{e}_5,\vec{e}_3)$$

From above, we know that $\begin{matrix}(A-2I):\vec{b}_4\to & \vec{b}_3\to\vec{0}\\ & \vec{b}_5\to\vec{0}\end{matrix}$ and $\begin{matrix}sp(\vec{e}_4,\vec{e}_5)=sp(\vec{b}_3,\vec{b}_5)\\ sp(\vec{e}_4,\vec{e}_5,\vec{e}_3)=sp(\vec{b}_3,\vec{b}_5,\vec{b}_4)\end{matrix}$

Since $\vec{b}_4\in null((A-2I)^2)$ and $\vec{b}_4\notin null(A-2I)$, we can pick $\vec{b}_4=\vec{e}_3$.

Let $\vec{b}_3=(A-\lambda I)\vec{b}_4=(A-2I)\vec{e}_3=-\vec{e}_5$.

Since $\vec{b}_5\in null(A-2I)$ and $\vec{b}_5\neq\vec{b}_3$, we can pick $\vec{b}_5=\vec{e}_4$.

$$V=\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5\end{bmatrix}=\begin{bmatrix}0 & -2+i & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 2 & -1 & 0 & 0\end{bmatrix}, J=\begin{bmatrix}i & 0 & 0 & 0 & 0\\ 0 & i & 0 & 0 & 0\\ 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 2\end{bmatrix}, A=VJV^{-1}$$

2. Find a Jordan canonical form and a Jordan basis for the matrix A

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Solution：

It is easy to find that the A only has the eigenvalue 2, whose algebraic multiplicity is 5.

$$A-2I = \begin{bmatrix} 0 & 5 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ nullity} = 2, \qquad null(A-2I) = sp(\vec{e}_1, \vec{e}_4)$$

$$(A-2I)^2 = \begin{bmatrix} 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ nullity} = 4, \qquad null((A-2I)^2) = sp(\vec{e}_1, \vec{e}_4, \vec{e}_2, \vec{e}_5)$$

$$(A-2I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ nullity} = 5, \qquad null((A-2I)^3) = sp(\vec{e}_1, \vec{e}_4, \vec{e}_2, \vec{e}_5, \vec{e}_3)$$

From above, we know that

$$(A-2I) : \begin{matrix} \vec{b}_3 \to & \vec{b}_2 \to \vec{b}_1 \to \vec{0} \\ & \vec{b}_5 \to \vec{b}_4 \to \vec{0} \end{matrix} \quad \text{and} \quad \begin{matrix} sp(\vec{e}_1, \vec{e}_4) = sp(\vec{b}_1, \vec{b}_4) \\ sp(\vec{e}_1, \vec{e}_4, \vec{e}_2, \vec{e}_5) = sp(\vec{b}_1, \vec{b}_4, \vec{b}_2, \vec{b}_5) \\ sp(\vec{e}_1, \vec{e}_4, \vec{e}_2, \vec{e}_5, \vec{e}_3) = sp(\vec{b}_1, \vec{b}_4, \vec{b}_2, \vec{b}_5, \vec{b}_3) \end{matrix}$$

Since $\vec{b}_3 \in null((A-2I)^3)$ and $\vec{b}_3 \notin null((A-2I)^2)$, we can pick $\vec{b}_3 = \vec{e}_3$.

Let $\vec{b}_2 = (A-\lambda I)\vec{b}_3 = (A-2I)\vec{e}_3 = \vec{e}_2$, and $\vec{b}_1 = (A-\lambda I)\vec{b}_2 = (A-2I)\vec{e}_2 = 5\vec{e}_1$.

Since $\vec{b}_5 \in null((A-2I)^2)$ and $\vec{b}_5 \notin null(A-2I)$ and $\vec{b}_5 \neq \vec{b}_3$, we can pick $\vec{b}_5 = \vec{e}_5$.

Let $\vec{b}_1 = (A-\lambda I)\vec{b}_5 = (A-2I)\vec{e}_5 = -\vec{e}_1 + \vec{e}_4$.

$$V = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, A = VJV^{-1}$$