姓名: SOLUTION 葉均承 應數一線性代數

學號: ________________ **Quiz 1** 考試日期: 2023/02/22

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Find the characteristic polynomial, the eigenvalues of the following matrix. Pick **one eigenvalue** to find a corresponding eigenvector.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & -3 \\ 5 & 0 & -1 \end{bmatrix}$$

Answer: (a) the characteristic polynomial: $-\lambda^3 - 5\lambda^2 + 2\lambda - 8 = (-1-\lambda)(2-\lambda)(4-\lambda)$.

(b) the eigenvalues and a corresponding eigenvectors: $(4, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}), (-1, \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}), (2, \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix})$.

2. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

used to generate the Fibonacci sequence.

Answer: the n^{th} Fibonacci number $F_n =$ $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

Solution：

It is easy to find out that the eigenvalues of A are $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Also, we have $\vec{v}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue λ_1 and $\vec{v}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue λ_2.

It is well know that $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2$. Therefore, we have

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\left(\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{5}}(\vec{v}_1 - \vec{v}_2)$$

Thus

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} &= A^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \frac{1}{\sqrt{5}} A^n (\vec{v}_1 - \vec{v}_2) = \frac{1}{\sqrt{5}}(A^n \vec{v}_1 - A^n \vec{v}_2) \\ &= \frac{1}{\sqrt{5}}(\lambda_1^n \vec{v}_1 - \lambda_2^n \vec{v}_2) = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^n \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} - (\frac{1-\sqrt{5}}{2})^n \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}\right) \end{aligned}$$

Therefore

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$