姓名: SOLUTION　　　　　　　　　　　　　　　　葉均承　　應數一線性代數

學號: ______________　　**Quiz 5**　　考試日期: 2023/03/21

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Let $W = sp([1,0,1,1],[1,1,1,0],[-1,1,0,1])$ is a subspace of $\mathbb{R}^4$. Using the Gram-Schmidt process to find an orthonormal basis for W. Given $\vec{b} = [3,2,0,1]$, please find the projection $\vec{b}_W$.

Answer: an orthonormal basis for W is $\{\frac{1}{\sqrt{3}}[1,0,1,1], \frac{1}{\sqrt{15}}[1,3,1,-2]\ \frac{1}{\sqrt{3}}[-1,1,0,1]\}$,

$\vec{b}_W =$ $[\frac{4}{\sqrt{3}}, \frac{7}{\sqrt{15}}, 0]$

Solution：

Let $\vec{a_1} = [1,0,1,1]$, $\vec{a_2} = [1,1,1,0]$, $\vec{a_3} = [-1,1,0,1]$,

$$\begin{aligned}\vec{v_1} &= [1,0,1,1],\\ \vec{q_1} &= \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{3}}[1,0,1,1],\end{aligned}$$

$$\begin{aligned}\vec{v_2} &= \vec{a_2} - \frac{\vec{a_2}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} = \vec{a_2} - \frac{2}{3}\vec{v_1} = \frac{1}{3}[1,3,1,-2],\\ \vec{q_2} &= \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{1}{\sqrt{15}}[1,3,1,-2],\end{aligned}$$

$$\begin{aligned}\vec{v_3} &= \vec{a_3} - \frac{\vec{a_3}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} - \frac{\vec{a_3}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2} = \vec{a_3} = [-1,1,0,1],\\ \vec{q_3} &= \frac{\vec{v_3}}{\|\vec{v_3}\|} = \frac{1}{\sqrt{3}}[-1,1,0,1],\end{aligned}$$

Let $\vec{b}_W = [b_1, b_2, b_3]$

$$b_1 = \vec{b}\cdot\vec{q_1} = \frac{4}{\sqrt{3}}$$

$$b_2 = \vec{b}\cdot\vec{q_2} = \frac{7}{\sqrt{15}}$$

$$b_3 = \vec{b}\cdot\vec{q_3} = 0$$

2. Given

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Factor A in the form $A = QR$, where Q is a matrix with orthonormal column vectors and R is an upper-triangular matrix.

Solution：

By previous problem, we have:

$$\begin{aligned} \vec{v_1} &= \vec{a_1}, \\ \vec{q_1} &= \frac{1}{\sqrt{3}} \vec{a_1}, \\ \vec{a_1} &= \sqrt{3}\, \vec{q_1} \end{aligned}$$

$$\begin{aligned} \vec{v_2} &= \vec{a_2} - \frac{2}{\sqrt{3}} \vec{q_1} \\ \vec{a_2} &= \frac{2}{\sqrt{3}} \vec{q_1} + \frac{\sqrt{5}}{\sqrt{3}} \vec{q_2} \end{aligned}$$

$$\begin{aligned} \vec{v_3} &= \vec{a_3}, \\ \vec{q_3} &= \sqrt{3}\, \vec{a_3}, \end{aligned}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{15}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{3}{\sqrt{15}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{15}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{15}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \; R = \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & 0 \\ 0 & \frac{\sqrt{5}}{\sqrt{3}} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$