109-1, quiz 1

2. (50%) Prove that the given relation holds for all vectors, matrices and scalars for which the expression are defined.

$$(AB)^T = B^T A^T$$

$m, n, s \in \mathbb{R}$

$\triangle$ $\underset{m\times n}{A = [a_{ij}]}$, $\underset{n\times s}{B = [b_{ij}]}$, let $AB = C = [c_{ij}]$, $m\times s$

Note $c_{ij} = \sum_k a_{ik} b_{kj}$ $\quad \therefore (AB)^T = C^T = [c'_{ij}]$ $\quad \therefore c'_{ij} = c_{ji} = \sum_k a_{jk} b_{ki}$

$\triangle$ $A^T = [a'_{ij}]$, $B^T = [b'_{ij}]$, let $B^T A^T = D = [d_{ij}]$

$a'_{ij} = a_{ji}$ $\qquad b'_{ij} = b_{ji}$

$d_{ij} = \sum_k b'_{ik} a'_{kj} = \sum_k b_{ki} a_{jk} = \sum_k a_{jk} b_{ki} = c'_{ij}$

$\therefore C = D$

常見錯誤: A, B 不只 2×2

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\quad B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},\quad AB=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & \\ & \end{bmatrix}$$

$$(AB)^T$$

$$A^T B^T$$

可行改短

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix},\quad B=\begin{bmatrix} b_{11} & \cdots & b_{1s} \\ \vdots & & \\ b_{n1} & \cdots & b_{ns} \end{bmatrix}$$

$$AB=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1n}b_{n1} & \cdots & a_{11}b_{1s}+a_{12}b_{2s}+\cdots+a_{1n}b_{ns} \\ \vdots & & \vdots \\ a_{m1}b_{11}+a_{m2}b_{21}\cdots+a_{mn}b_{n1} & \cdots & a_{m1}b_{1s}+a_{m2}b_{2s}+\cdots+a_{mn}b_{ns} \end{bmatrix}$$

ex:

solve $[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & -3 & 5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$ A is in r-e form

$\downarrow$

$0x + 0y + 0z = -1$ $\times$ $\therefore$ No solution!

ex:

solve $[A|\vec{b}] = \begin{array}{c} \begin{array}{ccccc} & r & & s & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 0 & -4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ A is in rref

Let $x_2 = r$, $x_4 = s$

$$\begin{cases} 1x_1 + -3x_2 + 0x_3 + 5x_4 + 0x_5 = -4 \leftarrow x_1 - 3x_2 + 5x_4 = -4 \Rightarrow x_1 - 3r + 5s = -4 \Rightarrow x_1 = -4 + 3r - 5s \\ 0x_1 + 0x_2 + 1x_3 + 2x_4 + 0x_5 = 7 \leftarrow x_3 + 2x_4 = 7 \Rightarrow x_3 + 2s = 7 \Rightarrow x_3 = 7 - 2s \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 = 1 \leftarrow x_5 = 1 \Rightarrow x_5 = 1 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0 \leftarrow 0 = 0 \end{cases}$$

solution: $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} -4+3r-5s \\ r \\ 7-2s \\ s \\ 1 \end{bmatrix}}_{\text{general solution!}} = \begin{bmatrix} -4 \\ 0 \\ 7 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5s \\ 0 \\ -2s \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 7 \\ 0 \\ 1 \end{bmatrix} + r\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

一般解 (通解)

$\therefore$ there's infinity many solution

Def.

$A\vec{x}=\vec{b}$: linear system

一致性

1. the system is consistent if it has one or more solution.
2. the system is inconsistent if it has no solution.

Thm

$A\vec{x}=\vec{b}$: linear system , $[A|\vec{b}] \sim [H|\vec{c}]$. H is in rref.

1. $A\vec{x}=\vec{b}$ is inconsistent.

   iff $[H|\vec{c}]$. has a row with all 0 at left but non-zero at the right part.

2. $A\vec{x}=\vec{b}$ is consistent and every column of H has a pivot

   $\Rightarrow$ unique solution

3. $A\vec{x}=\vec{b}$ is consistent and some columns of H has no pivot

   $\Rightarrow$ infinity many solution

Def

elementary matrix can be obtained by apply $\boxed{\text{one}}$ elementary row operation to an identity matrix.

ex:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + 4R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{2} R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

elementary matrix

Thm

A: m×n matrix , E: m×m elementary matrix

EA: apply the same elementary row operation from E to A

ex:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$A \ \Big\{ \ R_3 \to \tfrac{1}{2} R_3$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, \quad E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \frac{a_{31}}{2} & \frac{a_{32}}{2} & \frac{a_{33}}{2} & \frac{a_{34}}{2} \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21}+4a_{31} & a_{22}+4a_{32} & a_{23}+4a_{33} & a_{24}+4a_{34} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$A \ \underbrace{R_2 \to R_2 + 4R_3}$$

$$A \ \Big\{ \ R_1 \leftrightarrow R_2$$

$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

ex:

$$A \underset{E_1}{\overset{R_3 \to \frac{1}{2}R_3}{\sim}} A_1 \underset{E_2}{\overset{R_2 \to R_2 + 4R_3}{\sim}} A_2 \underset{E_3}{\overset{R_1 \leftrightarrow R_2}{\sim}} A_3 \underset{E_4}{\sim} A_4 \sim \cdots \underset{E_n}{\sim} A_n$$

$$A_1 = E_1 A$$

$$A_2 = E_2 A_1 = E_2 E_1 A$$

$$A_3 = E_3 A_2 = E_3 E_2 E_1 A$$

$$A_n = E_n E_{n-1} \cdots E_3 E_2 E_1 A$$

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ex:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 2 & 3 & -1 \\ 4 & 5 & -2 \end{bmatrix} \quad 1$$

$R_{(1)}$: $$\underset{}{\overset{R_1 \leftrightarrow R_2}{\sim}} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 4 & 5 & -2 \end{bmatrix} \quad 2$$

$$R_{(1)} \leftrightarrow E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_{(2)}$: $$\overset{R_3 \to R_3 - 2R_1}{\sim} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix} \quad 3$$

$$R_{(2)} \leftrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$1 \xrightarrow{R_{(1)}} 2 \xrightarrow{R_{(2)}} 3 \xrightarrow{R_{(3)}} 4$$

$R_{(3)}$: $$\overset{R_3 \to R_3 + R_2}{\sim} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix} = H \quad 4$$

$$R_{(3)} \leftrightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$H = R_{(3)}(R_{(2)}(R_{(1)}(A))), \qquad H = E_3 E_2 E_1 A$$

$$H = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix} \quad ④$$

$$\tilde{R}_{(3)}: \underbrace{R_3 \to R_3 - R_2}\quad \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix} \quad ③ \qquad \tilde{R}_{(3)} \leftrightarrow F_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad F_3 = E_3^{-1}$$

$$\tilde{R}_{(3)} \quad \tilde{R}_{(2)} \quad \tilde{R}_{(1)}$$

$$④ \to ③ \to ② \to ①$$

$$\tilde{R}_{(2)}: \underbrace{R_3 \to R_3 + 2R_1}\quad \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 4 & 5 & -2 \end{bmatrix} \quad ② \qquad \tilde{R}_{(2)} \leftrightarrow F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad F_2 = E_2^{-1}$$

$$\tilde{R}_{(1)}: \underbrace{R_1 \leftrightarrow R_2}\quad \begin{bmatrix} 0 & 1 & -3 \\ 2 & 3 & -1 \\ 4 & 5 & -2 \end{bmatrix} ① = A \qquad \tilde{R}_{(1)} \leftrightarrow F_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_1 = E_1^{-1}$$

$$A = \tilde{R}_{(1)}\left(\tilde{R}_{(2)}\left(\tilde{R}_{(3)}(H)\right)\right)$$

$$A = F_1 F_2 F_3 H = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1} H}$$

↑ 1-5 内容

1-5 Inverse matrix

- $a, b \in \mathbb{R}$ $\qquad ax = b \Rightarrow x = b/a \qquad 3x = 5 \Rightarrow x = 5/3$

$$ax = b$$

$$\left(\frac{1}{a}\right) ax = \left(\frac{1}{a}\right) b$$

$$\left(\frac{1}{a}\cdot a\right)x = 1\cdot x = x = b/a$$

$$\therefore \quad x = \frac{b}{a}$$

$$\frac{1}{a} = a^{-1}$$

- $A_{n\times n}, \ \vec{b}_{n\times 1}$

$$A\vec{x} = \vec{b}$$

$$C A\vec{x} = C\vec{b}$$

$$(CA)\vec{x} \overset{?}{=} I\vec{x} = \vec{x}$$

$$\therefore \vec{x} = C\vec{b}$$

Q. How to find C s.t. $CA = I$

Q: if $CA = I \Rightarrow AC \overset{?}{=} I$

Thm

A : $n \times n$ matrix

If $\exists\, C_{n\times n}$, $D_{n\times n}$ s.t. $CA = I_n$, $AD = I_n$

then $C = D$

p.f.

$$CAD = C(AD) = C \cdot I = C$$
$$\| \quad (CA)D = I \cdot D = D$$

$\therefore\ C = D$

Def

- $A_{n\times n}$: invertible if $\exists\, C_{n\times n}$ s.t. $CA = AC = I$

  Denote $C = A^{-1}$ which is the inverse of A

- $A_{n\times n}$: singular if A is NOT invertible

Thm

every elementary matrix is invertible.

p.f.

E : elementary matrix, if $\exists$ R : elementary row operation s.t. $E = R(I)$

$\exists$ R^{-1} : elementary row operation s.t. $R^{-1} \circ R$ = identity = $R \circ R^{-1}$

① $R : R_i \leftrightarrow R_j$, $R^{-1} = R_j \leftrightarrow R_i$

② $R : R_i \to r R_i$, $R^{-1} : R_i \to \frac{1}{r} R_i$

③ $R : R_i \to R_i + r \cdot R_j$, $R^{-1} : R_i \to R_i - r R_j$

Let $\breve{E} = R^{-1}(I)$

$\therefore$ $\breve{E} E = R^{-1}(E) = R^{-1}(R(I)) = I$

$E \breve{E} = R(\breve{E}) = R(R^{-1}(I)) = I$

$\therefore$ $\breve{E} = E^{-1}$

Thm

A, B : invertible $n \times n$ matrix

$\Rightarrow$ AB : invertible and $(AB)^{-1} = B^{-1}A^{-1}$

pf.

A, B : invertible $\Rightarrow \exists A^{-1}, B^{-1}$ s.t. $AA^{-1} = A^{-1}A = I$, $BB^{-1} = B^{-1}B = I$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A \cdot I \cdot A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1} \cdot I \cdot B = B^{-1}B = I$$

$$\therefore (AB)^{-1} = B^{-1}A^{-1}$$

Thm

The following are equivalent :

1. $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$
2. $A \sim I$ ← row equivalent
3. A : invertible

p.f. ② $\Rightarrow$ ③

$\because A \sim I$ $\therefore \exists R_{(1)}, R_{(2)}, \dots, R_{(k)}$: elementary row operation

s.t. $R_{(k)}(\cdots R_{(3)}(R_{(2)}(R_{(1)}(A))) \cdots) = I$

Let $E_i = R_{(i)}(I)$ $\therefore I = E_k \cdots E_3 E_2 E_1 A$ Recall: every elementary matrix is invertible.

$\therefore \forall i, \exists E_i^{-1}$: inverse of E_i

$$(E_1^{-1} E_2^{-1} \cdots E_k^{-1}) \cdot I = \underbrace{(E_1^{-1} \underbrace{E_2^{-1} \cdots \underbrace{E_k^{-1})(E_k}_{I} \cdots E_3 E_2}_{I} E_1}_{I} A) = A$$

$$\| $$

$$(E_1^{-1} E_2^{-1} \cdots E_k^{-1})$$

$\therefore A = (E_1^{-1} E_2^{-1} \cdots E_k^{-1})$ $\because (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) \cdot (E_k \cdots E_3 E_2 E_1) = I$

$(E_k \cdots E_3 E_2 E_1) \cdot (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) = I$

$\therefore A^{-1} = (E_k \cdots E_3 E_2 E_1)$

(3) $\Rightarrow$ (4)

A : invertible $\Rightarrow A^{-1}$: exist $\therefore \forall \vec{b}$ s.t. $A\vec{x} = \vec{b} \Rightarrow \underbrace{A^{-1}(A\vec{x})}_{= \vec{x}} = A^{-1}\vec{b}$

$\therefore \vec{x} = A^{-1}\vec{b}$ exists. (check $A\vec{x} = A(A^{-1}\vec{b}) = \vec{b}$)

$(1) \Rightarrow (2)$ $A \sim I$

$A\vec{x} = \vec{b}$ has a solution, $\text{rref}([A | \vec{b}]) = [H | \vec{c}]$

then $H = \text{rref}(A)$

① $H = I$ ✓

② $H \neq I$ i.e. $\exists$ row without pivot i.e. $\exists$ row: all 0's

$$\left[\begin{array}{cccccc|c} & & & & & & \vec{c} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \triangle \end{array}\right]$$

nonzero $\Rightarrow$ No solution X

△ How to find the inverse matrix of A?

$$[A | I] \underset{R_{(1)}}{\sim} [E_1 A | E_1] \underset{R_{(2)}}{\sim} [E_2 E_1 A | E_2 E_1] \sim \cdots$$

$$\cdots \underset{R_{(k)}}{\sim} [E_k \cdots E_2 E_1 A | E_k \cdots E_2 E_1] = [I | A^{-1}]$$

EXAMPLE 4 Determine whether the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix}$$

is invertible, and find its inverse if it is.

SOLUTION We have

$$\left[\begin{array}{rrr|rrr} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & -5 & 3 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 14 & -8 & -1 \\ 0 & 1 & 0 & -17 & 10 & 1 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right].$$

Therefore, A is an invertible matrix, and

$$A^{-1} = \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix}.$$