1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Find a Jordan canonical form and a Jordan basis for the matrix $A$

$$A = \begin{bmatrix} 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

**Solution：**

It is easy to find that the $A$ has the eigenvalue 3, whose algebraic multiplicity is 3 and $A$ has the eigenvalue 2, whose algebraic multiplicity is 3.

$$A - 3I = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \text{ i.e. nullity} = 2 \text{ and } null(A-3I) = sp(\vec{e}_1, \vec{e}_2).$$

$$(A - 3I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ i.e. nullity} = 3 \text{ and } null((A-3I)^2) = sp(\vec{e}_1, \vec{e}_2, \vec{e}_3).$$

From above, we know that

$$\begin{aligned} (A-3I) : \vec{b}_2 \to \vec{b}_1 &\to \vec{0} \\ \vec{b}_3 &\to \vec{0} \end{aligned}$$

Since $\vec{b}_3 \in null((A-3I)^2)$ and $\vec{b}_3 \notin null(A-3I)$, we pick $\vec{b}_2 = \vec{e}_3$, and $\vec{b}_1 = (A-3I)\vec{e}_3 = 2\vec{e}_1$.
Since $null(A-3I) = sp(\vec{e}_1, \vec{e}_2) = sp(\vec{b}_1, \vec{b}_3)$, we can pick $\vec{b}_3 = \vec{e}_2$.
Therefore,

$$\begin{aligned} (A-3I) : \vec{e}_3 \to 2\vec{e}_1 &\to \vec{0} \\ \vec{e}_2 &\to \vec{0} \end{aligned}$$

$$A-2I=\begin{bmatrix}1&0&2&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&4\\0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\end{bmatrix}\quad \text{i.e. nullity} = 2 \text{ and } null(A-2I)=sp(\vec{e}_4,\vec{e}_5).$$

$$(A-2I)^2=\begin{bmatrix}1&0&4&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&4\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}\quad \text{i.e. nullity} = 3 \text{ and } null((A-3I)^2)=sp(\vec{e}_4,\vec{e}_5,\begin{bmatrix}8\\0\\-4\\0\\0\\1\end{bmatrix})$$

From above, we know that

$$\begin{aligned}(A-2I):\vec{b}_5\to\vec{b}_4\to\vec{0}\\ \vec{b}_6\to\vec{0}\end{aligned}$$

Since $\vec{b}_5\in null((A-2I)^2)$ and $\vec{b}_5\notin null(A-2I)$, we can pick $\vec{b}_5=[8,\ 0,\ \ -4,\ 0,\ 0,\ 1]^T$.
Let $\vec{b}_4=(A-2I)\vec{b}_5=-\vec{e}_5$.
Since $\vec{b}_6\in null(A-2I)$ and $null(A-2I)=sp(\vec{b}_4,\vec{b}_6)$, we can pick $\vec{b}_6=\vec{e}_4$.

$$V=\begin{bmatrix}\vec{b}_1&\vec{b}_2&\vec{b}_3&\vec{b}_4&\vec{b}_5&\vec{b}_6\end{bmatrix}=\begin{bmatrix}2&0&0&0&8&0\\0&0&1&0&0&0\\0&1&0&0&-4&0\\0&0&0&0&0&1\\0&0&0&-1&0&0\\0&0&0&0&1&0\end{bmatrix},J=\begin{bmatrix}3&1&0&0&0&0\\0&3&0&0&0&0\\0&0&3&0&0&0\\0&0&0&2&1&0\\0&0&0&0&2&0\\0&0&0&0&0&2\end{bmatrix},A=VJV^{-1}$$