姓名: SOLUTION　　　　　　　　　　　　　　　　　　　　　　　　葉均承　　應數一線性代數

學號: ________　　　　　　　　**Quiz 2**　　　　　　　　考試日期: 2024/03/06

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Given a matrix A and use it to answer the following question. (a) find the eigenvalues and a corresponding eigenvectors of A. (b) find the algebraic multiplicity and the geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -7 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix}$$

Answer: (a) find the eigenvalues and a corresponding eigenvectors of A: $(2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$, $(-2, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix})$.

(b) Answer: **for eigenvalue -2, its alg. multiplicity is 1 where its geo. multiplicity is 1.**
for eigenvalue 2, its alg. multiplicity is 2 where its geo. multiplicity is 1.

(c) Is A diagonalizable? (Yes / $\boxed{No}$) .
If not, why? **for eigenvalue 2, its alg. multiplicity is 2 where its geo. multiplicity is 1.**

If so, find an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$.
C= ✗ , and D= ✗ .

Solution：

Since the characteristic polynomial of A is $p_A(\lambda) = \det(A - \lambda I) = -\lambda^3 + 2\lambda^2 + 4\lambda - 8 = (-2-\lambda)(2-\lambda)^2$, the algebraic multiplicity of 2 is 2 and the algebraic multiplicity of -2 is 1.

$$A - 2I \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow null(A-2I) = sp(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) \Rightarrow \text{the geometric multiplicity of 2 is 1.}$$

$$A - (-2)I \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow null(A-2I) = sp(\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}) \Rightarrow \text{the geometric multiplicity of -2 is 1.}$$

2. Let A be an $n \times n$ matrix. (下面兩題挑一題證即可，記得註明你要證哪題)
   - (a) Prove that if A is similar to rA where r is a real scalar other than 1 or -1, then all eigenvalues of A are zero. [*Hint:* by 5-2 prob. 18. similar matrices have the same eigenvalues with the same algebraic multiplicities.]
   - (b) What can you say about A if it is diagonalizable and similar to rA for some r where $|r| \neq 1$?

Solution：

(a) If $r = 0$, trivial case!

If $r \neq 0$ and $|r| > 1$. Let λ_1 is an eigenvalue (possible complex) of A of maximum magnitude and there exist $\vec{v}_1 \neq \vec{0}$ so that $A\vec{v}_1 = \lambda_1 \vec{v}_1$. Thus $(rA)\vec{v}_1 = (r\lambda_1)\vec{v}_1$ and $r\lambda_1$ is an eigenvalue of rA. Since A is similar to rA and use the idea of 5-2 prob. 18, we know that $r\lambda_1$ is also an eigenvalue of A. However, $|r\lambda_1| > |\lambda_1|$. $(\Rightarrow\Leftarrow)$

If $r \neq 0$ and $|r| < 1$. Let $\tilde{\lambda}_1$ is an eigenvalue (possible complex) of A of minimum magnitude. Similarly, we have $|r\tilde{\lambda}_1| < |\tilde{\lambda}_1|$. $(\Rightarrow\Leftarrow)$

(b) $A = O_{n \times n}$.