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Quiz 3

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1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Let the sequence $a_0, a_1, a_2, \ldots$ be given by $a_0 = 2, a_1 = -1$ and $a_k = \frac{1}{2}a_{k-1} + \frac{1}{2}a_{k-2}$ for $k \geq 2$. Find the explicit formula for a_k.

   Answer: $a_k = 0.5^{k-1}$

   Solution：

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}, \det(A - \lambda I) = (0.5-\lambda)(-\lambda) - 0.5 = (1-\lambda)(-0.5-\lambda)$$

Therefore, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -0.5$.

$\boxed{\lambda_1 = 1}$,

$$rref(A - \lambda_1 I) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\boxed{\lambda_2 = -0.5}$,

$$rref(A - \lambda_2 I) = \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Method 1:

Let

$$C = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = A^k \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = (CDC^{-1})^k \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{2^k} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Method 2: Let

$$\begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = r_1\vec{v}_1 + r_2\vec{v}_2$$

Then

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = A^k \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = A^k(r_1\vec{v}_1 + r_2\vec{v}_2) = r_1\lambda_1^k\vec{v}_1 + r_2\lambda_2^k\vec{v}_2$$

Notice that $r_1 = 0, r_2 = 1$

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = A^k \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = 0.5^k \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.5^k \\ 0.5^{k-1} \end{bmatrix}$$

2. Solve the given system.

$$\begin{cases} x_1' = & x_1 & -3x_2 & +3x_3, \\ x_2' = & & -5x_2 & +6x_3, \\ x_3' = & & -3x_2 & +4x_3 \end{cases}$$

Answer: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1e^t \\ k_2e^t \\ k_3e^{-2t} \end{bmatrix} = \begin{bmatrix} k_1e^t + k_3e^{-2t} \\ k_2e^t + 2k_3e^{-2t} \\ k_2e^t + k_3e^{-2t} \end{bmatrix},$

Solution：

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

Such that $AC = CD$, and $\vec{x}' = A\vec{x}$

$$\text{Solve } \vec{y}' = D\vec{y}, \text{ i.e. } \begin{cases} y_1' = & y_1 \\ y_2' = & y_2 \\ y_3' = & -2y_3 \end{cases} \Rightarrow \begin{cases} y_1 = & k_1e^t \\ y_2 = & k_2e^t \\ y_3 = & k_3e^{-2t} \end{cases}$$

$$\vec{x} = C\vec{y} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1e^t + k_3e^{-2t} \\ k_2e^t + 2k_3e^{-2t} \\ k_2e^t + k_3e^{-2t} \end{bmatrix}$$