Quiz 6

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Given $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T([x, y, z]) = \frac{1}{7}[2x - 3y + 6z, 3x + 6y + 2z, -6x + 2y - 3z]$.

   (1) Determine whether the linear transformation T is orthogonal.

   Answer: **T is NOT orthogonal**

   (2) Find the angle between $T([1, 0, 1])$ and $T([1, 1, -1])$

   Answer: $\cos^{-1}(\frac{12}{\sqrt{170}\sqrt{99}})$

   Solution：

$$T([x, y, z])^T = \frac{1}{7}\begin{bmatrix} 2x - 3y + 6z \\ 3x + 6y + 2z \\ -6x + 2y - 3z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & -3 & 6 \\ 3 & 6 & 2 \\ -6 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

   Therefore, the standard matrix representation of T is

$$A = \frac{1}{7}\begin{bmatrix} 2 & -3 & 6 \\ 3 & 6 & 2 \\ -6 & 2 & -3 \end{bmatrix}$$

   Since $A^T A = \frac{1}{49}\begin{bmatrix} 49 & 0 & 36 \\ 0 & 49 & -12 \\ 36 & -12 & 49 \end{bmatrix}$, T is Not orthogonal!

$$T([1, 0, 1]) = \left(A * \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)^T = \frac{1}{7}[8, 5, -9]$$

$$T([1, 1, -1]) = \left(A * \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)^T = \frac{1}{7}[-7, 7, -1]$$

   the angle between $\vec{a} = T([1, 0, 1])$ and $\vec{b} = T([1, 1, -1])$ is

$$\cos^{-1}(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\ |\vec{b}|}) = \cos^{-1}(\frac{12}{\sqrt{170}\sqrt{99}})$$

2. Given $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T([x, y, z]) = \frac{1}{7}[2x - 3y + 6z, 3x + 6y + 2z, -6x + 2y + 3z]$.

(1) Determine whether the linear transformation T is orthogonal.

Answer: **T is orthogonal**

(2) Find the angle between $T([1, 0, 1])$ and $T([1, 1, -1])$

Answer: $\frac{\pi}{2}$

Solution：

$$T([x,y,z])^T = \frac{1}{7}\begin{bmatrix} 2x - 3y + 6z \\ 3x + 6y + 2z \\ -6x + 2y + 3z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & -3 & 6 \\ 3 & 6 & 2 \\ -6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Therefore, the standard matrix representation of T is

$$A = \frac{1}{7}\begin{bmatrix} 2 & -3 & 6 \\ 3 & 6 & 2 \\ -6 & 2 & 3 \end{bmatrix}$$

Since $A^T A = I$, T is orthogonal!

By textbook p. 356, underlineTheorem of Orthogonal Linear Transformation:
The angle between $T(\vec{x})$ and $T(\vec{y})$ equals the angle between $\vec{x}$ and $\vec{y}$.

Let the $\vec{x} = [1, 0, 1]$, $\vec{y} = [1, 1, -1]$. Since $\vec{x}$ and $\vec{y}$ are obviously perpendicular, $T(\vec{x})$ and $T(\vec{y})$ are also perpendicular. Hence, the angle between $T(\vec{x})$ and $T(\vec{y})$ is $\frac{\pi}{2}$.

3. Show that the real eigenvalues of an orthogonal matrix must be equal to 1 or -1.

Solution：

Section 6-3 #27.

4. Show that orthogonal matrices preserve the dot product of vectors. (i.e. $(A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y})$.)

Solution：

Theorem 6.6 (1).