Def.

$A\vec{x}=\vec{b}$: linear system

1. the system is consistent if it has one or more solutions
2. the system is inconsistent if it has no solution

Thm.

$A\vec{x}=\vec{b}$: linear system , $[A|\vec{b}] \sim [H|\vec{c}]$, where H: r-e form

1. $A\vec{x}=\vec{b}$ is inconsistent
   iff $[H|\vec{c}]$ has a row with all 0 in the left part but non-zero in the right part
2. $A\vec{x}=\vec{b}$ is consistent and every column of H has a pivot
   $\Rightarrow$ unique solution.
3. $A\vec{x}=\vec{b}$ is consistent and some column of H has no pivot
   $\Rightarrow$ infinitely many solution, with as many free variables as the number of pivot-free column in H.

Def.

elementary matrix can be obtained by apply one elementary row operation to an identity matrix.

ex:

$$\begin{tikzcd}
I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \arrow[r, "R_1 \leftrightarrow R_2"] \arrow[dr, "R_2 \to R_2 + 4R_3"'] \arrow[ddr, "R_3 \to \frac{1}{2}R_3"'] & \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \arrow[dr, dash] & \\
& \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix} \arrow[r, dash] & \text{elementary matrix} \\
& \begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{2}\end{bmatrix} \arrow[ur, dash] &
\end{tikzcd}$$

Thm.

$A_{m\times n}$: matrix , $E_{m\times m}$: elementary matrix

EA = apply the same elementary row operation from E to A.

ex :

$R_3 \to \frac{1}{2}R_3$ $R_2 \to R_2+4R_3$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \frac{a_{31}}{2} & \frac{a_{32}}{2} & \frac{a_{33}}{2} \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+4a_{31} & a_{22}+4a_{32} & a_{23}+4a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

ex:

$$A \overset{\overset{E_1}{R_1 \leftrightarrow R_2}}{\sim} A_1 \overset{\overset{E_2}{R_2 \to R_2+3R_1}}{\sim} A_2 \overset{E_3}{\sim} A_3 \sim \cdots A_n$$

$$A_1 = E_1 A$$

$$A_2 = E_2 A_1 = E_2 E_1 A$$

$$A_3 = E_3 A_2 = E_3 E_2 E_1 A$$

ex:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 2 & 3 & -1 \\ 4 & 5 & -2 \end{bmatrix} \quad R_{(1)}: \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 4 & 5 & -2 \end{bmatrix}$$

$$R_2 \leftrightarrow R_1$$

$$R_{(1)} \leftrightarrow E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{(2)}: \overset{R_3 \to R_3 - 2R_1}{\sim} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_1$$

$$R_{(2)} \leftrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$R_{(3)}: \overset{R_3 \to R_3 + R_2}{\sim} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix} = H$$

$$R_3 \to R_3 - R_2$$

$$R_{(3)} \leftrightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{cases} H = R_{(3)}(R_{(2)}((R_{(1)}(A)))) \\ H = E_3 E_2 E_1 A \end{cases}$$

$$A = R_{(1)}^{-1}(R_{(2)}^{-1}(R_{(3)}^{-1}((H)))) \quad \text{or} \quad A = E_1^{-1} E_2^{-1} E_3^{-1} H$$

$\boxed{1-5}$

- $a, b \in \mathbb{R}$,

$ax = b \Rightarrow x = b/a$

$3x = 5 \Rightarrow x = 5/3$

$ax = b$

$$\left[\frac{1}{a}a\right]x = \left[\frac{1}{a}\right]ax = \left[\frac{1}{a}\right]b$$

$$x = \frac{b}{a}$$

$\therefore x = \frac{1}{a}b$

- $A_{n\times n}$, $\vec{b}_{n\times 1}$,

$A\vec{x} = \vec{b}$

$C(A\vec{x}) = C\vec{b}$

$\vec{x} = I\vec{x}$

$\Rightarrow \vec{x} = C\vec{b}$

Q : How to find C s.t. $CA = I$

Q : if $CA = I \Rightarrow AC \overset{?}{=} I$

Thm

$A_{n\times n}$: matrix

If $\exists\ C_{n\times n},\ D_{n\times n}$ s.t. $AC = I$, $DA = I$.

then $C = D$

pf.

$$\begin{aligned} &DAC = D(AC) = D\cdot I = D \qquad\qquad \therefore\ C = D \\ &\quad\ \ \| \\ &\quad\ (DA)C = I\cdot C = C \end{aligned}$$

Def.

- $A_{n\times n}$: invertible if $\exists\ C_{n\times n}$ s.t. $CA = AC = I$. Denote $\overset{\substack{\text{inverse of } A \\ \Downarrow}}{C}$ by A^{-1}
- $A_{n\times n}$: singular if A is NOT invertible

① $R_i \leftrightarrow R_j \rightsquigarrow R_j \leftrightarrow R_i$

② $R_i \to R_i + rR_j \rightsquigarrow R_i \to R_i - rR_j$

③ $R_i \to rR_i \rightsquigarrow R_i \to \frac{1}{r} R_i$

Thm.

every elementary matrix is invertible

p.f.

E : elementary matrix if $\exists$ R: elementary row operation

s.t. $E = R(I)$

$\exists$ R^{-1} : elementary row operation s.t. $R^{-1} \circ R$ = identity

let $\tilde{E} = R^{-1}(I)$

$\therefore \tilde{E} \cdot E = R^{-1}(E) = R^{-1}(R(I)) = I$

$E\tilde{E} = R(\tilde{E}) = R(R^{-1}(I)) = I$

$\therefore \tilde{E} = E^{-1}$

ex: $E_3 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: $R : R_1 \rightarrow R_1 + 4R_3$ $\therefore$ $R^{-1} : R_1 \rightarrow R_1 - 4R_3$

$E_3^{-1} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thm

A, B : invertible $n \times n$ matrix

$\Rightarrow$ AB : invertible and $(AB)^{-1} = B^{-1}A^{-1}$

p.f.

A, B : invertible $\Rightarrow$ $\exists A^{-1}, B^{-1}$ s.t. $AA^{-1} = A^{-1}A = I$, $BB^{-1} = B^{-1}B = I$

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A \cdot I \cdot A^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$

$\therefore$ $(AB)^{-1} = B^{-1}A^{-1}$

Thm $A : n\times n$

The following are equivalent:

1. $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}$
2. $A \sim I$
3. A : invertible

p.f. (2) ⇒ (3)

$A\sim I \Rightarrow R_{(k)}(\cdots R_{(2)}(R_{(1)}(A))\cdots) = I$

let $E_i = R_{(i)}(I)$ $\therefore I = E_k\cdots\cdots E_2E_1A$

$$(E_1^{-1}E_2^{-1}\cdots E_k^{-1})\cdot I = (E_1^{-1}E_2^{-1}\cdots E_k^{-1})E_k\cdots\cdots E_2E_1A = A$$

$\therefore A = E_1^{-1}E_2^{-1}\cdots E_k^{-1} = (E_k\cdots E_2E_1)^{-1}$, $A^{-1} = E_k\cdots E_2E_1$

★ $[A\,|\,I] \overset{R_{(1)}}{\sim} [E_1A\,|\,E_1] \overset{R_{(2)}}{\sim} [E_2E_1A\,|\,E_2E_1] \sim \cdots$

$\cdots \overset{R_{(k)}}{\sim} [E_k\cdots E_2E_1A\,|\,E_k\cdots E_2E_1] = [I\,|\,A^{-1}]$

(3) ⇒ (1)

A : invertible iff A^{-1} : exist

$\forall \vec{b}$ s.t. $A\vec{x}=\vec{b} \Rightarrow \underset{\vec{x}}{\underset{\|}{A^{-1}(A\vec{x})}} = A^{-1}\vec{b}$

$\therefore \vec{x}=A^{-1}\vec{b}$

(check : $A(A^{-1}\vec{b}) = \vec{b}$)

1. $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}$
2. $A \sim I$

(1) ⇒ (2)

reduced row-echelon form

$A\vec{x}=\vec{b}$ has a solution ⇒ $\text{rref}([A|\vec{b}]) = [H|\vec{c}]$

then $H = \text{rref}(A)$

① $H = I$ ✓

② $H \neq I$ ∵ each column: elt above & below pivots are 0's ∴ each col has at most one pivot

∴ # pivots $\leq n$

∵ $H \neq I$ ∴ # pivots $< n$

∴ $H \neq I$ ⇒ some row has no pivot

i.e. row = all 0's

∴ the bottom row of H is $\vec{0}^T$

∵ $A \sim H$ ∴ $\exists\ E_1, E_2, \ldots, E_k$: elementary matrices s.t. $H = E_k \cdots E_2 E_1 A$

pick $\vec{b} = (E_k \cdots E_2 E_1)^{-1}\vec{e}_n$, where $\vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$

then $[A|\vec{b}] \sim [H|\vec{e}_n] = \left[\begin{array}{ccccccc|c} & & & * & & & & 0 \\ & & & & & & & 0 \\ & & & & & & & \vdots \\ & & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$ No Solution *

$A_{n\times n}$: invertible if $\exists\, C_{n\times n}$ s.t. $CA = AC = I$. Denote $\overset{\text{inverse of } A}{\overset{\downarrow}{C}}$ by A^{-1}

if $\exists\, C$ s.t $AC = I$

or $CA = I$

Thm

A, C : $n\times n$ matrices

$\Rightarrow$ $CA = I$ iff $AC = I$

p.f.

Assume $AC = I$, $\forall\, \vec{b}$, sol: $A\vec{x} = \vec{b}$, then $\vec{x} = C\vec{b}$ is a solution

Check: $A\vec{x} = AC\vec{b} = I\vec{b} = \vec{b}$ ✓

$\therefore \forall\, \vec{b}$, $A\vec{x} = \vec{b}$ has a solution $\therefore A \sim I$

$\therefore \exists\, E_k \cdots E_2 E_1$: elementary matrices s.t. $I = E_k \cdots E_2 E_1 A$

let $D = E_k \cdots E_2 E_1$ $\therefore \begin{cases} AC = I \\ DA = I \end{cases}$ $\therefore C = D$ $\therefore CA = I$

ex:

$A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$, find A^{-1}

sol.

$$\left[\begin{array}{cc|cc} 2 & 9 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & 9 & 1 & 0 \end{array}\right] \overset{R_2 \to R_2 - 2R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\overset{R_1 \to R_1 - 4R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & -4 & 9 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}$$

$$\boxed{A^{-1} \text{ exist} \Leftrightarrow [A \mid I] \sim [I \mid A^{-1}]}$$

Thm $A : n\times n$

The following are equivalent:

1. $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}\in\mathbb{R}^n$
2. $A \sim I$
3. A : invertible
4. $\exists\, E_1, E_2, \dots, E_k$: elementary matrices s.t. $A = E_1 E_2 \cdots E_k$
5. let $\vec{a}_i$ = the i^{th} column vector of A, then $sp(\vec{a}_1,\dots,\vec{a}_n)=\mathbb{R}^n$

p.f.

(5) $\Leftrightarrow$ (1)

$$A = \begin{bmatrix} \downarrow & \downarrow & & \downarrow \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$\underset{\text{has sol.}}{A\vec{x}=\vec{b}} \Leftrightarrow \vec{b} = \begin{bmatrix} \downarrow & \downarrow & & \downarrow \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n}_{\uparrow\ \text{linear comb. of } \{\vec{a}_i\}} \in sp(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$$

1-6

$$A\vec{x}=\vec{0}$$

Def.

a linear system ($A\vec{x}=\vec{b}$) is homogeneous if $\vec{b}=\vec{0}$

△ $A\vec{x}=\vec{b}$ has solution $\vec{x}_1 \neq \vec{x}_2$

$\Rightarrow A\vec{x}_1=\vec{b}=A\vec{x}_2 \Rightarrow A\vec{x}_1-A\vec{x}_2=A(\vec{x}_1-\vec{x}_2)=\vec{b}-\vec{b}=\vec{0}$

$\therefore A\vec{x}=\vec{0}$ has non-trivial solution ($\neq \vec{0}$)

Thm $A: nx$

☆ the sol set of $A\vec{x}=\vec{0}$ is a subspace of $\mathbb{R}^n$

$A\vec{x}=\vec{0}$ has solutions $\vec{h}_1, \vec{h}_2$

$\Rightarrow \forall r,s \in \mathbb{R}$, $r\vec{h}_1 + s\vec{h}_2$ still a solution for $A\vec{x}=\vec{0}$

p.f.

$A\vec{h}_1 = \vec{0}$, $A\vec{h}_2 = \vec{0}$

$\forall r,s \in \mathbb{R}$, $A(r\vec{h}_1 + s\vec{h}_2) = A(r\vec{h}_1) + A(s\vec{h}_2) = rA\vec{h}_1 + sA\vec{h}_2$

$= r\vec{0} + s\vec{0} = \vec{0} + \vec{0} = \vec{0}$

Def.

W is a subspace of $\mathbb{R}^n$

if ① W : subset of $\mathbb{R}^n$

② (i) closed under vector addition : $\forall \vec{u}, \vec{v} \in W \Rightarrow \vec{u}+\vec{v} \in W$

(ii) closed under scalar multiplication : $\forall r \in \mathbb{R}, \forall \vec{u} \in W \Rightarrow r\vec{u} \in W$

W:

☆ the sol set of $A\vec{x}=\vec{0}$ is a subspace of $\mathbb{R}^n$

Given A: n×n matrix.   Let $W=\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\}$

$\Rightarrow$ W: subspace of $\mathbb{R}^n$

ex: $W=\{[x,2x] \mid x\in\mathbb{R}\} \overset{\substack{\text{subset}\\ \downarrow}}{\subset} \mathbb{R}^2$

check: ②(i) $\forall\, a, b\in\mathbb{R}$, let $\vec{u}=[a,2a]$, $\vec{v}=[b,2b]$

$\vec{u}+\vec{v}=[a,2a]+[b,2b]=[a+b,\ 2a+2b]=[a+b,\ 2(a+b)]\in W$

②(ii) $\forall\, r$, $r\vec{u}=r[a,2a]=[ra,\ 2(ra)]\in W$

$\therefore$ W: subspace of $\mathbb{R}^2$ △ W is Not a subspace of $\mathbb{R}^3$ (since W is Not a subset of $\mathbb{R}^3$)

ex: $W=\{\vec{0}\}\subset\mathbb{R}^n$

Check: ②(i) $\vec{0}+\vec{0}=\vec{0}\in W$ $\therefore$ W: subspace of $\mathbb{R}^n$

②(ii) $\forall\, r\in\mathbb{R}$, $r\vec{0}=\vec{0}\in W$

ex: $W = \{ [a, 2a+1] \mid a \in \mathbb{R} \} \subset \mathbb{R}^2$

check: ②(i) $\forall a, b \in \mathbb{R}$, let $\vec{u} = [a, 2a+1]$, $\vec{v} = [\vec{b}, 2b+1]$

$$\vec{u} + \vec{v} = [a+b, \underset{\underset{2(a+b)+1}{\nparallel}}{2(a+b)+2}] \notin W$$

$\therefore$ W is Not a subspace of $\mathbb{R}^2$

Thm

Given $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \in \mathbb{R}^n$, let $W = sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$

$\Rightarrow$ W : subspace of $\mathbb{R}^n$

p.f.

① W : subset of $\mathbb{R}^n$

② (i) $\vec{u}, \vec{v} \in W$

let $\vec{u} = r_1 \vec{w}_1 + r_2 \vec{w}_2 + \ldots + r_k \vec{w}_k$, where $r_1, \ldots, r_k, s_1, \ldots, s_k \in \mathbb{R}$

$\vec{v} = s_1 \vec{w}_1 + s_2 \vec{w}_2 + \ldots + s_k \vec{w}_k$

$\vec{u} + \vec{v} = (r_1 + s_1)\vec{w}_1 + (r_2 + s_2)\vec{w}_2 + \ldots + (r_k + s_k)\vec{w}_k \in W$

(ii) $\forall \alpha \in \mathbb{R}$

$\alpha \vec{u} = \alpha (r_1 \vec{w}_1 + r_2 \vec{w}_2 + \ldots + r_k \vec{w}_k)$

$= (\alpha r_1) \vec{w}_1 + (\alpha r_2) \vec{w}_2 + \ldots + (\alpha r_k) \vec{w}_k \in W$

$\therefore$ W : subspace of $\mathbb{R}^n$

Def. $A_{m\times n}$

1. The nullspace of matrix $A_{m\times n}$ is $\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\}$ = null(A)

2. The row space of A = sp(row vectors of A) $\subset \mathbb{R}^n$ = row(A)

3. The column space of A = sp(column vectors of A) $\subset \mathbb{R}^m$ = col(A)

Moreover, nullspace, row space are subspaces of $\mathbb{R}^n$, column space is subspace of $\mathbb{R}^m$.

ex:

$$A = \begin{bmatrix} \textcircled{1} & 0 & \overset{\downarrow r}{3} \\ 0 & \textcircled{1} & -1 \end{bmatrix} \qquad A\vec{x}=\vec{0} \Rightarrow \begin{cases} x_1+3r=0 \\ x_2-1r=0 \end{cases} \Rightarrow \begin{cases} x_1=-3r \\ x_2=r \\ x_3=r \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

1. nullspace of A = $\mathrm{sp}\left(\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right) \subset \mathbb{R}^3 \longrightarrow \mathrm{sp}\left(\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}\right)$ or $\mathrm{sp}\left(\begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix}\right)$

2. row space of A = sp([1, 0, 3], [0, 1, -1]) $\subset \mathbb{R}^3$

3. column space of A = $\mathrm{sp}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) \subset \mathbb{R}^2$

Thm.

$\vec{p}$ is particular solution of $A\vec{x}=\vec{b}$

then 「$\vec{v}$ is a solution of $A\vec{x}=\vec{b}$」 homo. sol of $A\vec{x}=\vec{b}$

iff 「$\vec{v}$ has the form $\vec{p}+\vec{h}$, where $\vec{h}$ is a solution of $A\vec{x}=\vec{0}$」

pf.

(⇐)

$$A(\vec{p}+\vec{h}) = A\vec{p}+A\vec{h} = \vec{b}+\vec{0} = \vec{b}$$

(⇒)

$$\left.\begin{cases} A\vec{v}=\vec{b} \\ A\vec{p}=\vec{b} \end{cases}\right] \Rightarrow \underset{\vec{b}-\vec{b}=\vec{0}}{\underset{\parallel}{A\vec{v}-A\vec{p}}} = A(\vec{v}-\vec{p})$$

$$\therefore \vec{v} = \vec{p}+\underbrace{(\vec{v}-\vec{p})}_{\vec{h}}$$

Thm. $A_{n\times n}$

A: invertible

iff $A\vec{x}=\vec{0}$ has only one solution (only has the trivial s.l $\vec{0}$)

iff if $A\vec{x}=\vec{b}$ has a solution, then the solution is unique.

iff $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}\in\mathbb{R}^n$

pf.

if A: invertible, $A(A^{-1}\vec{b})=\vec{b}$

$\therefore \vec{x}=A^{-1}\vec{b}$ is a solution

if $\vec{v}$ is another solution, $A^{-1}(\ulcorner A\vec{v}=\vec{b}\lrcorner) \Rightarrow \vec{v}=A^{-1}\vec{b}=\vec{x}$

ex

$$A=\begin{bmatrix}2&0&3\\0&2&-1\end{bmatrix},\qquad A\vec{x}=\begin{bmatrix}3\\4\end{bmatrix}\quad\Rightarrow\quad\begin{bmatrix}2&0&3\\0&2&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$$

check $\begin{bmatrix}3/2\\2\\0\end{bmatrix}$ ← particular sol

is a sol of $A\vec{x}=\begin{bmatrix}3\\4\end{bmatrix}$

$$A=\begin{bmatrix}2&0&3\\0&2&-1\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&\overset{\downarrow r}{3/2}\\0&\textcircled{1}&-1/2\end{bmatrix}\Rightarrow\begin{cases}x_1+\frac{3}{2}r=0\\x_2-\frac{1}{2}r=0\\x_3=r\end{cases}\quad\therefore\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\frac{3}{2}r\\-\frac{1}{2}r\\r\end{bmatrix}=\begin{array}{c}\frac{1}{2}r\begin{bmatrix}3\\-1\\2\end{bmatrix}\\ \| \\ r\begin{bmatrix}\frac{3}{2}\\-\frac{1}{2}\\1\end{bmatrix}\end{array}$$

$\therefore$ the nullspace of A is $sp\left(\begin{bmatrix}\frac{3}{2}\\-1/2\\1\end{bmatrix}\right)$ or $sp\left(\begin{bmatrix}3\\-1\\2\end{bmatrix}\right)$

$\therefore$ all sol of $A\vec{x}=\begin{bmatrix}3\\4\end{bmatrix}$ are $\begin{bmatrix}3/2\\2\\0\end{bmatrix}+\vec{h}$ where $\vec{h}\in sp\left(\begin{bmatrix}3\\-1\\2\end{bmatrix}\right)$

all sol of $A\vec{x}=\begin{bmatrix}3\\4\end{bmatrix}$ are $\begin{bmatrix}3/2\\2\\0\end{bmatrix}+r\begin{bmatrix}3\\-1\\2\end{bmatrix}$, where $r\in\mathbb{R}$

the sol set of $A\vec{x}=\begin{bmatrix}3\\4\end{bmatrix}$ is $\left\{\begin{bmatrix}3/2\\2\\0\end{bmatrix}+r\begin{bmatrix}3\\-1\\2\end{bmatrix}\,\middle|\,r\in\mathbb{R}\right\}$

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix} \qquad \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right]$$

Thm

$A_{m\times n}$

(1) if $A\vec{x}=\vec{b}$ has a solution, then the solution is unique

iff (2) $H=\operatorname{rref}(A)$, $H=\left[\begin{array}{c} I_{n\times n} \\ \hline O_{m-n,n} \end{array}\right]$ $\swarrow m \geq n$

pf.

(2) ⇒ (1)

$$[A|\vec{b}] \sim [H|\vec{c}] = \left[\begin{array}{c|c} I & \vec{c} \\ \hline 0 & \end{array}\right]$$ ⇒ the solution is unique if it exists.

(1) ⇒ (2)

every column of H has a pivot. $\#(\text{pivot}) = n$

every row of H has at most one pivot ⇒ $m \geq n$

$A=[a_{ij}]$, say pivots are $a_{1\sigma(1)}, a_{2\sigma(2)}, \dots, a_{n\sigma(n)}$

$1 \leq \sigma(1) < \sigma(2) < \sigma(3) < \cdots < \sigma(n) \leq n \Rightarrow \sigma(j) = j$

Cor.

$A_{m\times n}$, $m<n$ (m：方程數， n：變數個數)

$\Rightarrow$ (1) $A\vec{x}=\vec{0}$ has infinitely many solution.

(2) $A\vec{x}=\vec{b}$ has a solution $\Rightarrow$ the solution is NOT unique.

Recall A : $m\times n$

1. If $A\vec{x}=\vec{b}$ has a solution $\Leftrightarrow$ $\vec{b}\in col(A)$ ← the column space of A
2. the column space of A ($col(A)$) is a subspace of $\mathbb{R}^n$

   and the # of free variable is equal to the # of non-pivot column of $rref(A)$