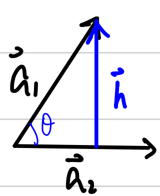
$\boxed{4-4}$

Def. The n-box in $\mathbb{R}^m$ determined by n indep. vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is the set S

$$S = \{ \vec{x} \in \mathbb{R}^m \mid \vec{x} = t_1\vec{a}_1 + t_2\vec{a}_2 + \cdots + t_n\vec{a}_n, \ \forall i, \ 0 \le t_i \le 1 \}$$

$n=1$, the volume of 1-box is $\|\vec{a}_1\| = \sqrt{\vec{a}_1 \cdot \vec{a}_1} = \sqrt{\det(\vec{a}_1^T \vec{a}_1)}$

$n=2$, the volume of 2-box is $\|\vec{h}\| \times \|\vec{a}_2\|$

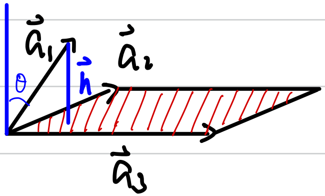


$$\begin{aligned} \text{vol}^2 &= \|\vec{h}\|^2 \times \|\vec{a}_2\|^2 = \|\vec{a}_1\|^2 \sin^2\theta \, \|\vec{a}_2\|^2 = \|\vec{a}_1\|^2 \|\vec{a}_2\|^2 (1-\cos^2\theta) \\ &= \|\vec{a}_1\|^2 \|\vec{a}_2\|^2 - \cos^2\theta \, \|\vec{a}_1\|^2 \|\vec{a}_2\|^2 = (\vec{a}_1 \cdot \vec{a}_1)(\vec{a}_2 \cdot \vec{a}_2) - (\vec{a}_1 \cdot \vec{a}_2)(\vec{a}_2 \cdot \vec{a}_1) \\ &= \det\left(\begin{bmatrix} (\vec{a}_1 \cdot \vec{a}_1) & (\vec{a}_1 \cdot \vec{a}_2) \\ (\vec{a}_2 \cdot \vec{a}_1) & (\vec{a}_2 \cdot \vec{a}_2) \end{bmatrix} \right) = \det\left(\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \end{bmatrix} \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \right) \end{aligned}$$

$$\text{vol} = \sqrt{\det\left(\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \end{bmatrix} \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \right)} \underset{\substack{\uparrow \\ \text{by 4-1 in } \mathbb{R}^3}}{=} \|\vec{a}_1 \times \vec{a}_2\|$$

$\boxed{m=3}$

$n=3$



$$\text{vol} = \|\vec{h}\| \times (\text{vol of 2-box determined by } \vec{a}_2, \vec{a}_3)$$

in $\mathbb{R}^3$

$$= \|\vec{h}\| \times \|\vec{a}_2 \times \vec{a}_3\| = \|\vec{a}_1\| \cos\theta \, \|\vec{a}_2 \times \vec{a}_3\|$$

$$\text{vol}^2 = \|\vec{a}_1\|^2 \cos^2\theta \, \|\vec{a}_2 \times \vec{a}_3\|^2 = \left(\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)\right)^2 = \det\left(\begin{bmatrix} - \vec{a}_1^T - \\ - \vec{a}_2^T - \\ - \vec{a}_3^T - \end{bmatrix}\right) \det\left(\begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} - \vec{a}_1^T - \\ - \vec{a}_2^T - \\ - \vec{a}_3^T - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix}\right)$$

$$\therefore \text{vol} = \sqrt{\det\left(\begin{bmatrix} - \vec{a}_1^T - \\ - \vec{a}_2^T - \\ - \vec{a}_3^T - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix}\right)}$$

☆

vol. of n-box determined by $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$

$= (\underline{\text{Altitude}} \ \|\vec{h}\|) \times (\text{vol. of } (n-1)\text{-box determined by } \vec{a}_2, \ldots, \vec{a}_n)$

高

Lemma

Given $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \in \mathbb{R}^n$. Let $A = \begin{bmatrix} \vert & \vert & \vert & & \vert \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \\ \vert & \vert & \vert & & \vert \end{bmatrix}$, $B = \begin{bmatrix} \vert & \vert & \vert & & \vert \\ \vec{b} & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \\ \vert & \vert & \vert & & \vert \end{bmatrix}$

where $\vec{b} = \vec{a}_1 - r_2\vec{a}_2 - r_3\vec{a}_3 - \cdots - r_n\vec{a}_n$ for some scalar $r_2, r_3, \dots, r_n$

$\Rightarrow \det(A^TA) = \det(B^TB)$

p.f.

Let E_i is the elementary matrix with the operator 「$R_1 \to R_1 - r_i R_i$」

then $B^T = E_n E_{n-1} \cdots E_3 E_2 A^T = EA^T$, where $E = E_n E_{n-1} \cdots E_3 E_2$

$\therefore B = (EA^T)^T = AE^T$

$\therefore \det(B^TB) = \det((EA^T)(AE^T)) = \underline{\det(E)}\det(A^TA)\underline{\det(E^T)} = \underline{1} \cdot \det(A^TA) \cdot \underline{1} = \det(A^TA)$

Recall Property 5 in $\boxed{4\text{-}2}$

PROPERTY 5 The Row-Addition Property

If the product of one row of a square matrix A by a scalar is added to a different row of A, the determinant of the resulting matrix is the same as $\det(A)$.

Thm 4.7

The volume of the n-box in $\mathbb{R}^m$ determined by n indep. vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ is given by

$$\text{vol} = \sqrt{\det(A^T A)}, \quad \text{where } A = \begin{bmatrix} | & | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \\ | & | & | & & | \end{bmatrix}$$

p.f.

By induction! (歸納)

1. $n = 1, 2$ checked!

2. Assume the the thm is proved for all k-box for $k \le n-1$

3. When there's n vectors, let $\vec{a}_1 = \vec{b} - \vec{p}$ s.t. $\vec{p} \in sp(\vec{a}_2, \vec{a}_3, \dots, \vec{a}_n)$ and $\vec{b} \perp \vec{a}_i$, $\forall i = 2 \sim n$

↳ by the idea of projection or by the Gram-Schmidt process in Ch6.

$\therefore$ we have $\vec{p} = r_2\vec{a}_2 + r_3\vec{a}_3 + \dots + r_n\vec{a}_n$ for some scalar $r_2, r_3, \dots, r_n$ and $\vec{b} = \vec{a}_1 - (r_2\vec{a}_2 + \dots + r_n\vec{a}_n)$

Let $B = \begin{bmatrix} | & | & | & & | \\ \vec{b} & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \\ | & | & | & & | \end{bmatrix}$, $B^T B = \begin{bmatrix} - & \vec{b}^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_n^T & - \end{bmatrix} \begin{bmatrix} | & | & | & & | \\ \vec{b} & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{a}_2 & \vec{b}\cdot\vec{a}_3 & \cdots & \vec{b}\cdot\vec{a}_n \\ \vec{a}_2\cdot\vec{b} & \vec{a}_2\cdot\vec{a}_2 & \vec{a}_2\cdot\vec{a}_3 & \cdots & \vec{a}_2\cdot\vec{a}_n \\ \vec{a}_3\cdot\vec{b} & \vec{a}_3\cdot\vec{a}_2 & & \ddots & \\ \vdots & \vdots & & & \\ \vec{a}_n\cdot\vec{b} & \vec{a}_n\cdot\vec{a}_2 & & & \vec{a}_n\cdot\vec{a}_n \end{bmatrix}$

$$\therefore\ B^TB=\begin{bmatrix}\vec b\cdot\vec b & \vec b\cdot\vec a_2 & \vec b\cdot\vec a_3 & \cdots & \vec b\cdot\vec a_n\\ \vec a_2\cdot\vec b & \vec a_2\cdot\vec a_2 & \vec a_2\cdot\vec a_3 & \cdots & \vec a_2\cdot\vec a_n\\ \vec a_3\cdot\vec b & \vec a_3\cdot\vec a_2 & \ddots & & \\ \vdots & \vdots & & & \\ \vec a_n\cdot\vec b & \vec a_n\cdot\vec a_2 & & & \vec a_n\cdot\vec a_n\end{bmatrix}=\begin{bmatrix}\vec b\cdot\vec b & 0 & 0 & \cdots & 0\\ 0 & \vec a_2\cdot\vec a_2 & \vec a_2\cdot\vec a_3 & \cdots & \vec a_2\cdot\vec a_n\\ 0 & \vec a_3\cdot\vec a_2 & \ddots & & \\ \vdots & \vdots & & & \\ 0 & \vec a_n\cdot\vec a_2 & & & \vec a_n\cdot\vec a_n\end{bmatrix}$$

$$\therefore\ \det(B^TB)=\|\vec b\|^2\times\det\left(\begin{bmatrix}\vec a_2\cdot\vec a_2 & \vec a_2\cdot\vec a_3 & \cdots & \vec a_2\cdot\vec a_n\\ \vec a_3\cdot\vec a_2 & \ddots & & \\ \vdots & & & \\ \vec a_n\cdot\vec a_2 & & & \vec a_n\cdot\vec a_n\end{bmatrix}\right)$$

by induction

$$=\underbrace{\|\vec b\|^2}_{\text{高}}\times(\text{vol. of }(n-1)\text{-box determined by }\vec a_2,\dots,\vec a_n)^2=(\text{volume of }n\text{-box})^2$$

by Lemma

$$\det(B^TB)\ \|\ \det(A^TA)$$

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相關 排序

Note the volume of n-box is Not relevant with order of vectors

Similar with the prove of Lemma, but use the Property 2

PROPERTY 2 The Row-Interchange Property

If two different rows of a square matrix A are interchanged, the determinant of the resulting matrix is $-\det(A)$.

Let $\underset{\displaystyle EA^T}{\overset{\shortparallel}{C^T}}$ is change the order of some rows in $A^T = \begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_n^T & - \end{bmatrix}$

$\therefore \det(E) = (-1)^k$ for some k

$\det(C^TC) = \det(E)\det(A^TA)\det(E^T) = (-1)^k\det(A^TA)(-1)^k = \det(A^TA)$