$\boxed{3-3}$

$(V, \oplus, \otimes)$

V : vector space , $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$: ordered basis for V

$\forall \vec{v} \in V$, $\exists\, r_1, r_2, \ldots, r_n$: coefficient (scalar) s.t. $\vec{V} = r_1\vec{b}_1 + r_2\vec{b}_2 + \ldots + r_n\vec{b}_n$

$\vec{v}_B = [r_1, r_2, \ldots, r_n]$

Thm

1. $(\vec{v} \oplus \vec{u})_B = \vec{v}_B + \vec{u}_B \quad \forall \vec{u}, \vec{v} \in V$

2. $(r \otimes \vec{u})_B = r(\vec{u}_B) \quad \forall r \in \mathbb{R}$

$\boxed{3-4}$

$T : V \longrightarrow V'$: linear transformation , $(V, \oplus, \otimes)$, $(V', \oplus', \otimes')$: vector space

<u>Def 1</u> if $\begin{cases} ① \ T(\vec{u}) \oplus' T(\vec{v}) = T(\vec{u} \oplus \vec{v}) & \forall \vec{u}, \vec{v} \in V \\ ② \ r \otimes' T(\vec{u}) = T(r \otimes \vec{u}) & \forall r : \text{scalar} \end{cases}$

<u>Def 2</u> if $(r \otimes' T(\vec{u})) \oplus' (s \otimes' T(\vec{v})) = T((r \otimes \vec{u}) \oplus (s \otimes \vec{v}))$ $\quad \forall \vec{u}, \vec{v} \in V$, $\forall r, s$: scalar

☆ kernel of T = $\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$ → is the zero vector in V'

☆ $A_3.\ \vec{0} \oplus' \vec{v} = \vec{v}$

Thm

$T(\vec{0}_v) = \vec{0}_{v'}$ ← is the zero vector in V'

↑ is the zero vector in V

Thm V, V': vector space

$T: V \to V'$: linear transformation , Given $B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$: basis for V

$\Rightarrow \forall \vec{v} \in V$, $T(\vec{v})$ is uniquely determined by $T(\vec{b}_1), T(\vec{b}_2), \dots, T(\vec{b}_n)$

↳ i.e. 給定 $T(\vec{b}_1), T(\vec{b}_2), \dots, T(\vec{b}_n)$ 的值後, $T(\vec{v})$ 的值也確定了

↳ T 就是唯一的一個 linear transformation

Thm

A linear transformation T ($V \to V'$) is one-to-one iff $\ker(T) = \{\vec{0}_v\}$

↳ i.e. $\forall\ \vec{u} \neq \vec{v} \in V \Leftrightarrow T(\vec{u}) \neq T(\vec{v})$

Def. $T: V \longrightarrow V'$: invertible linear transformation

$\exists\, \tilde{T}: V' \to V$: linear transformation s.t. $(\tilde{T} \circ T)(\vec{v}) = \vec{v}$, $\forall\, \vec{v} \in V$

$(T \circ \tilde{T})(\vec{u}') = \vec{u}'$, $\forall\, \vec{u}' \in V'$

Denote T^{-1} ~~if A is the s.m.r. of T and A is invertible~~

Thm. $T: V \longrightarrow V'$: invertible linear transformation

iff T is one-to-one and onto V'

↳ if $T(\vec{v}_1) = T(\vec{v}_2)$ in $V' \Rightarrow \vec{v}_1 = \vec{v}_2$ $\forall\, \vec{v}' \in V'$, $\exists\, \vec{v} \in V$ s.t. $T(\vec{v}) = \vec{v}'$

Cor $T: V \longrightarrow V'$: invertible linear transformation $\Rightarrow T^{-1}: V' \longrightarrow V$

$\vec{v} \longmapsto T(\vec{v})$ $T(\vec{v}) \longmapsto \vec{v}$

Def. $T: V \longrightarrow V'$: isomorphism

if T is invertible linear transformation (one-to-one and onto V')

Thm

$\hat{T} : V \to \mathbb{R}^n$, where $\dim(V) = n$, $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$: ordered basis for V

$\vec{v} \mapsto \vec{v}_B$

$\Rightarrow$ $\hat{T}$ is an invertible linear transformation (isomorphism)

B : ordered basis for V $\qquad$ B' : ordered basis for V'

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\hat{T}"'] & V' \arrow[d] \\
\mathbb{R}^n \arrow[r, "\tilde{T}"] & \mathbb{R}^m
\end{tikzcd}
\qquad
\begin{tikzcd}
\vec{v} \arrow[r, mapsto] \arrow[d, mapsto] & T(\vec{v}) \arrow[d, mapsto] \\
\vec{v}_B \arrow[r, mapsto] & T(\vec{v})_{B'}
\end{tikzcd}$$

$T : V \to V'$: linear transformation

$$\vec{v}_B \longmapsto \tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$$
$$\overset{\shortparallel}{A\vec{v}_B} \xrightarrow{\text{Def.}} A \text{ is the matrix representation of } T \text{ relative to } B, B'$$

or A is the s.m.r. of $\tilde{T}$

$B = (\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n)$ $\qquad\qquad$ B'

$$
\begin{tikzcd}
V \arrow[rrr, "T"] \arrow[ddd] & & & V' \arrow[ddd] \\
 & \vec{v} \arrow[r, mapsto] \arrow[d, mapsto] & T(\vec{v}) \arrow[d, mapsto] & \\
 & \vec{v}_B \arrow[r, mapsto] & T(\vec{v})_{B'} & \\
\mathbb{R}^n \arrow[rrr, "\tilde{T}"] & & & \mathbb{R}^m
\end{tikzcd}
$$

$$\vec{v}_B \overset{A}{\longmapsto} \tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$$
$$\tilde{T}(\vec{v}_B) = A\vec{v}_B$$

$\bigstar$ $\tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$

$$A = \begin{bmatrix} | & | & & | \\ \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \cdots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

$$\vec{v}_B = \vec{e}_j \iff \vec{v} = \vec{b}_j$$

$$\therefore \tilde{T}(\vec{e}_j) = \tilde{T}((\vec{b}_j)_B) = T(\vec{b}_j)_{B'}$$

$$\therefore A = \begin{bmatrix} | & | & & | \\ \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \cdots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ T(\vec{b}_1)_{B'} & T(\vec{b}_2)_{B'} & \cdots & T(\vec{b}_n)_{B'} \\ | & | & & | \end{bmatrix}$$

Def. $T : V \to V'$: linear transformation

$\exists$ A is the <u>matrix representation of T relative to B, B'</u>

s.t. $\forall \vec{v} \in V$, $T(\vec{v})_{B'} = A\vec{v}_B$

7-2

$B = (\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n)$ $\qquad$ B'

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d] & V' \arrow[d] \\ \mathbb{R}^n \arrow[r, "\tilde{T}"] & \mathbb{R}^m \end{tikzcd}$$

$$\begin{tikzcd} \vec{v} \arrow[r, mapsto] \arrow[d, mapsto] & T(\vec{v}) \arrow[d, mapsto] \\ \vec{v}_B \arrow[r, mapsto] & T(\vec{v})_{B'} \end{tikzcd}$$

$$\vec{v}_B \xmapsto{R_{BB'}} \tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$$

$$\tilde{T}(\vec{v}_B) = R_{BB'}\vec{v}_B$$

$$R_{B,B'} = \begin{bmatrix} | & | & & | \\ \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \cdots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

$$\vec{v}_B = \vec{e}_j \iff \vec{v} = \vec{b}_j$$

$$\therefore \tilde{T}(\vec{e}_j) = \tilde{T}((\vec{b}_j)_B) = T(\vec{b}_j)_{B'}$$

$$R_{B,B'} = \begin{bmatrix} | & | & & | \\ \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \cdots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ T(\vec{b}_1)_{B'} & T(\vec{b}_2)_{B'} & \cdots & T(\vec{b}_n)_{B'} \\ | & | & & | \end{bmatrix}$$

☆ $\tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$

Def. $T: V \to V'$: linear transformation

$\exists\, R_{B,B'}$ is the matrix representation of T relative to B, B'

s.t. $\forall \vec{v} \in V$, $T(\vec{v})_{B'} = R_{B,B'}\vec{v}_B$

3-5 Inner Product Space

Recall $(V, \oplus, \otimes)$ is a vector space if $A_0 \sim A_4$, $S_0 \sim S_4$ holds

$\forall \vec{u}, \vec{v}, \vec{w} \in V$, $\forall r, s \in \mathbb{R}$

A_0 : $\vec{v} \oplus \vec{u} \in V$

A_1 : $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$

A_2 : $\vec{u} \oplus \vec{w} = \vec{w} \oplus \vec{u}$

A_3 : $\vec{0} \oplus \vec{v} = \vec{v}$

A_4 : $\vec{v} \oplus (-\vec{v}) = (-\vec{v}) \oplus \vec{v} = \vec{0}$

S_0 : $r \otimes \vec{v} \in V$

S_1 : $r \otimes (\vec{u} \oplus \vec{v}) = r \otimes \vec{u} \oplus r \otimes \vec{v}$

S_2 : $(r+s) \otimes \vec{v} = (r \otimes \vec{v}) \oplus (s \otimes \vec{v})$

S_3 : $r \otimes (s \otimes \vec{v}) = (rs) \otimes \vec{v}$

S_4 : $1 \otimes \vec{v} = \vec{v}$

$\langle\, , \rangle : V \times V \longrightarrow \mathbb{R}$

$\vec{u}, \vec{v} \longmapsto \langle \vec{u}, \vec{v} \rangle$

Def. $(V, \oplus, \otimes, \langle\, , \rangle)$ is an inner product space if $(V, \oplus, \otimes)$ is a vector space and $D_1 \sim D_4$ holds

$\forall \vec{u}, \vec{v}, \vec{w} \in V$, $\forall r, s$: scalar

D_1 : $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

+ in $\mathbb{R}$

D_2 : $\langle \vec{u}, \vec{v} \oplus \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

D_3 : $r \langle \vec{u}, \vec{v} \rangle = \langle r \otimes \vec{u}, \vec{v} \rangle = \langle \vec{u}, r \otimes \vec{v} \rangle$

D_4 : $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}_V$

Def. $(V, \oplus, \otimes, \langle , \rangle)$ is an inner product space
$\forall \vec{v} \in V$, the magnitude or the norm of $\vec{v}$ is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

Prop. $\forall \vec{u}, \vec{v} \in V$, the angle between $\vec{u}$ and $\vec{v}$ is $\cos^{-1}\left(\dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \times \|\vec{v}\|} \right)$

Def. $\forall \vec{u}, \vec{v} \in V$, $\vec{u}, \vec{v}$ are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$

Thm (Schwarz Inequality): $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \times \|\vec{v}\|$

Thm (Triangle Inequality): $\|\vec{u} \oplus \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Recall $(P, \oplus, \otimes)$ is a vector space, where P is the set of all polynomials with real coefficient.

$\oplus, \otimes$ are normal operator for polynomials

ex: $(P^{[0,1]}, \oplus, \otimes, \langle , \rangle)$ is an inner product space

, where $P^{[0,1]}$ is the set of all polynomials with real coefficient and domain $0 \leq x \leq 1$

$\oplus, \otimes$ are normal operator for polynomials

$\langle f(x), g(x) \rangle = \int_0^1 f(x) g(x)\, dx \quad , \quad \forall f(x), g(x) \in P^{[0,1]}$

ex: $(P^{[a,b]}, \oplus, \otimes, \langle , \rangle_w)$ is an inner product space

, where $P^{[a,b]}$ is the set of all polynomials with real coefficient and domain $a \leq x \leq b$

$\oplus, \otimes$ are normal operator for polynomials

$\langle f(x), g(x) \rangle_w = \int_a^b f(x) g(x) w(x)\, dx, \quad \forall f(x), g(x) \in P^{[a,b]}$