姓名: SOLUTION 葉均承 應數一線性代數

學號: ________________

Quiz 12

考試日期: 2025/05/21

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Find all $a, b \in \mathbb{C}$ such that the matrix A is unitarily diagonalizable.

$$A = \begin{bmatrix} a & i \\ i & b \end{bmatrix}$$

Answer: $a =$____ , $b =$____

Solution：

Check A is normal matrix and use the Theroem 9.7.

$$A^*A = \begin{bmatrix} \overline{a} & -i \\ -i & \overline{b} \end{bmatrix} \begin{bmatrix} a & i \\ i & b \end{bmatrix} = \begin{bmatrix} a\overline{a}+1 & i\overline{a}-bi \\ -ai+i\overline{b} & b\overline{b}+1 \end{bmatrix} = \begin{bmatrix} a\overline{a}+1 & i(\overline{a}-b) \\ i(\overline{b}-a) & b\overline{b}+1 \end{bmatrix}$$

$$AA^* = \begin{bmatrix} a & i \\ i & b \end{bmatrix} \begin{bmatrix} \overline{a} & -i \\ -i & \overline{b} \end{bmatrix} = \begin{bmatrix} a\overline{a}+1 & -ia+\overline{b}i \\ \overline{a}i-ib & b\overline{b}+1 \end{bmatrix} \begin{bmatrix} a\overline{a}+1 & i(\overline{b}-a) \\ i(\overline{a}-b) & b\overline{b}+1 \end{bmatrix}$$

Thus, we know that A is normal if $(\overline{a}-b) = (\overline{b}-a)$.

$$(\overline{a}-b) = (\overline{b}-a) \ \Rightarrow (\overline{a}+a) = (\overline{b}+b)$$

Since $(\overline{a}+a)$ is twice the real part of a and $(\overline{b}+b)$ is twice the real part of b. The condition holds when a, b has the same real part.

2. Prove that, if A is normal and B is unitarily equivalent to A, then B is normal.
   Note: A and B is unitarily equivalent if there is a unitary matrix U such that $B = U^{-1}AU$.

   Solution：

   Section 9.3 problem 25 (b) (上課有證).

3. Please provide **(and explain)** a square matrix A that A is unitarily diagonalizable but NOT Hermitian.

   Solution：

   counterexample for Section 9.3 problem 19 (d).