

不可使用手機、計算器，禁止作弊!

- Find a Jordan canonical form for A , where $(A - 3I)$ has nullity 2, $(A - 3I)^2$ has nullity 3, $(A - 3I)^3$ has nullity 4, $(A - 3I)^k$ has nullity 5 for $k \geq 4$; $(A + I)^K$ has nullity 1 for $k \geq 1$; $(A - 2I)$ has nullity 2, $(A - 2I)^2$ has nullity 4 for $k \geq 2$.

Solution :

Answer:

$(A - 3I)$ has nullity 2,

$(A - 3I)^2$ has nullity 3,

$(A - 3I)^3$ has nullity 4,

$(A - 3I)^k$ has nullity 5 for $k \geq 4$

$$\Rightarrow (A - 3I) : \begin{array}{l} \vec{e}_1 \rightarrow \vec{0} \\ \vec{e}_5 \rightarrow \vec{e}_4 \rightarrow \vec{e}_3 \rightarrow \vec{e}_2 \rightarrow \vec{0} \end{array}$$
$$\vec{e}_5 \rightarrow \vec{e}_4 \rightarrow \vec{e}_3 \rightarrow \vec{e}_2 \rightarrow \vec{0}$$

$(A + I)^K$ has nullity 1 for $k \geq 1$

$$\Rightarrow (A + I) : \vec{e}_6 \rightarrow \vec{0}$$

$(A - 2I)$ has nullity 2,

$(A - 2I)^2$ has nullity 4 for $k \geq 2$

$$\Rightarrow \quad (A - 2I) : \quad \begin{array}{l} \vec{e}_8 \rightarrow \vec{e}_7 \rightarrow \vec{0} \\ \vec{e}_9 \rightarrow \vec{e}_{10} \rightarrow \vec{0} \end{array}$$
$$\vec{e}_9 \rightarrow \vec{e}_{10} \rightarrow \vec{0}$$

A diagram of a sparse matrix with non-zero entries highlighted in red boxes. The matrix is represented by a grid of cells. The non-zero entries are:

- Row 1, Column 1: 3
- Row 2, Column 1: 3
- Row 2, Column 2: 1
- Row 2, Column 3: 0
- Row 2, Column 4: 0
- Row 3, Column 1: 0
- Row 3, Column 2: 3
- Row 3, Column 3: 1
- Row 3, Column 4: 0
- Row 4, Column 1: 0
- Row 4, Column 2: 0
- Row 4, Column 3: 3
- Row 4, Column 4: 1
- Row 5, Column 1: 0
- Row 5, Column 2: 0
- Row 5, Column 3: 0
- Row 5, Column 4: 3

The matrix is symmetric, with non-zero entries only on the main diagonal and the first off-diagonal.

2. Find a Jordan canonical form and a Jordan basis for the given matrix.

$$A = \begin{bmatrix} 2 & 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Solution :

The characteristic poly is $(2 - \lambda)^2(-1 - \lambda)^3$, then the eigenvalue are 2 and -1.

$$\begin{aligned} rref(A - 2I) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad null(A - 2I) = sp\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) \\ rref((A - 2I)^2) &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad null((A - 2I)^2) = sp\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) \\ rref(A + I) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad null(A + I) = sp\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ rref((A + I)^2) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad null((A + I)^2) = sp\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) \end{aligned}$$

$$(A - 2I) : \vec{b}_2 \rightarrow \vec{b}_1 \rightarrow \vec{0}, \quad (A + I) : \begin{matrix} \vec{b}_4 \rightarrow \vec{b}_3 \rightarrow \vec{0} \\ \vec{b}_5 \rightarrow \vec{0} \end{matrix}$$

$$\text{Pick } \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ then } \vec{b}_1 = (A - 2I)\vec{b}_2 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\text{Pick } \vec{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ then } \vec{b}_3 = (A + I)\vec{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \text{ Thus } \vec{b}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} | & | & | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We have

$$C^{-1}AC = J$$