姓名: SOLUTION 葉均承 應數一線性代數

學號: ______________

Quiz 13

考試日期: 2025/05/28

不可使用手機、計算器，禁止作弊!

1. Find a Jordan canonical form for A, where $(A-3I)$ has nullity 2, $(A-3I)^2$ has nullity 3, $(A-3I)^3$ has nullity 4, $(A-3I)^k$ has nullity 5 for $k \geq 4$; $(A+I)^K$ has nullity 1 for $k \geq 1$; $(A-2I)$ has nullity 2, $(A-2I)^2$ has nullity 4 for $k \geq 2$.

Solution：

Answer:

$$\left.\begin{array}{l} (A-3I) \text{ has nullity } 2, \\ (A-3I)^2 \text{ has nullity } 3, \\ (A-3I)^3 \text{ has nullity } 4, \\ (A-3I)^k \text{ has nullity } 5 \text{ for } k \geq 4 \end{array}\right. \Rightarrow \quad (A-3I): \begin{array}{r} \vec{e}_1 \to \vec{0} \\ \vec{e}_5 \to \vec{e}_4 \to \vec{e}_3 \to \vec{e}_2 \to \vec{0} \end{array}$$

$$(A+I)^K \text{ has nullity } 1 \text{ for } k \geq 1 \quad \Rightarrow \quad (A+I): \ \vec{e}_6 \to \vec{0}$$

$$\left.\begin{array}{l} (A-2I) \text{ has nullity } 2, \\ (A-2I)^2 \text{ has nullity } 4 \text{ for } k \geq 2 \end{array}\right. \Rightarrow \quad (A-2I): \begin{array}{l} \vec{e}_8 \to \vec{e}_7 \to \vec{0} \\ \vec{e}_9 \to \vec{e}_{10} \to \vec{0} \end{array}$$

$$\begin{bmatrix} \boxed{3} & & & & \\ & \begin{array}{|cccc|} \hline 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \\ \hline \end{array} & & \Large 0 & \\ & & \boxed{-1} & & \\ & \Large 0 & & \begin{array}{|cc|} \hline 2 & 1 \\ 0 & 2 \\ \hline \end{array} & \\ & & & & \begin{array}{|cc|} \hline 2 & 1 \\ 0 & 2 \\ \hline \end{array} \end{bmatrix}$$

2. Find a Jordan canonical form and a Jordan basis for the given matrix.

$$A = \begin{bmatrix} 2 & 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Solution：

The characteristic poly is $(2-\lambda)^2(-1-\lambda)^3$, then the eigenvalue are 2 and -1.

$$rref(A-2I) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; null(A-2I) = sp(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix})$$

$$rref((A-2I)^2) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; null((A-2I)^2) = sp(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix})$$

$$rref(A+I) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; null(A+I) = sp(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix})$$

$$rref((A+I)^2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; null((A+I)^2) = sp(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix})$$

$$(A-2I): \; \vec{b}_2 \to \vec{b}_1 \to \vec{0} \; , \quad (A+I): \; \begin{matrix} \vec{b}_4 \to \vec{b}_3 \to \vec{0} \\ \vec{b}_5 \to \vec{0} \end{matrix}$$

Pick $\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, then $\vec{b}_1 = (A-2I)\vec{b}_2 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Pick $\vec{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, then $\vec{b}_3 = (A+I)\vec{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Thus $\vec{b}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$$C = \begin{bmatrix} | & | & | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We have

$$C^{-1}AC = J$$