姓名: SOLUTION　　　　　　　　　　　　葉均承　　應數一線性代數

學號: ________________　　**Quiz 5**　　考試日期: 2025/03/26

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Let $W = sp([0,1,-1,-1],[1,-1,1,0],[-1,1,0,1])$ is a subspace of $\mathbb{R}^4$. Using the Gram-Schmidt process to find an orthonormal basis for $W$ and then transform this to an orthonormal basis for $\mathbb{R}^4$. Given $\vec{b} = [-1,3,2,0]$, please find the projection $\vec{b}_W$.

   Answer: an orthonormal basis for $W$ is $\{\frac{1}{\sqrt{3}}[0,1,-1,-1], \frac{1}{\sqrt{15}}[3,-1,1,-2], \frac{1}{\sqrt{15}}[1,3,2,1]\}$

   $\vec{b}_W =$ $[0,3,1,1]$

   an orthonormal basis for $\mathbb{R}^4$ is $\{\frac{1}{\sqrt{3}}[0,1,-1,-1], \frac{1}{\sqrt{15}}[3,-1,1,-2], \frac{1}{\sqrt{15}}[1,3,2,1], \frac{1}{\sqrt{3}}[1,0,-1,1]\}$

**Solution：**

Let $\vec{a_1} = [0,1,-1,-1]$, $\vec{a_2} = [1,-1,1,0]$, $\vec{a_3} = [-1,1,0,1]$,

$$\vec{v_1} = [0,1,-1,-1], \qquad \vec{q_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{3}}[0,1,-1,-1],$$

$$\vec{v_2} = \vec{a_2} - \frac{\vec{v_1}\cdot\vec{a_2}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} = \frac{1}{3}[3,-1,1,-2], \qquad \vec{q_2} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{1}{\sqrt{15}}[3,-1,1,-2]$$

$$\vec{v_3} = \vec{a_3} - \frac{\vec{v_1}\cdot\vec{a_3}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} - \frac{\vec{v_2}\cdot\vec{a_3}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2} = \frac{1}{5}[1,3,2,1], \qquad \vec{q_3} = \frac{\vec{v_3}}{\|\vec{v_3}\|} = \frac{1}{\sqrt{15}}[1,3,2,1],$$

Let $\vec{a}_4 = [1,0,0,0]$, $\vec{a}_5 = [0,1,0,0]$, $\vec{a}_6 = [0,0,1,0]$, $\vec{a}_7 = [0,0,0,1]$

$$\begin{aligned}\vec{v_4} &= \vec{a}_4 - \frac{\vec{v_1}\cdot\vec{a_4}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} - \frac{\vec{v_2}\cdot\vec{a_4}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2} - \frac{\vec{v_3}\cdot\vec{a_4}}{\vec{v_3}\cdot\vec{v_3}}\vec{v_3} = \frac{1}{3}[1,0,-1,1],\\ \vec{q_4} &= \frac{\vec{v_4}}{\|\vec{v_4}\|} = \frac{1}{\sqrt{3}}[1,0,-1,1],\end{aligned}$$

$\boxed{\text{Method 1:}}$

$$\begin{aligned}\vec{b}_W &= (\vec{b}\cdot\vec{q_1})\ \vec{q_1} + (\vec{b}\cdot\vec{q_2})\ \vec{q_2} + (\vec{b}\cdot\vec{q_3})\ \vec{q_3}\\ &= [0,3,1,1]\end{aligned}$$

$\boxed{\text{Method 2:}}$

$$\vec{b}_W = \vec{b} - \vec{b}_{W^\perp} = \vec{b} - (\vec{q_4}\cdot\vec{b})\vec{q_4} = [0,3,1,1]$$