Quiz 6

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Given a matrix A,

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

(a) Find an orthogonal diagonalization of A, that is, find an orthogonal matrix C such that $C^{-1}AC$ is a diagonal matrix D.

(b) The eigenvalues of A are $\lambda = 0$ and $\lambda = 3$ (repeated). Let W be the eigenspace corresponding to $\lambda = 3$. Found the projection matrix P onto W.

Answer: $C = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $P = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Solution：

(a) First, we find the eigenvectors for each eigenvalue.

For $\lambda = 0$: $A - 0I = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. $\vec{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

For $\lambda = 3$: $A - 3I = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow .\ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Now we apply the Gram-Schmidt process to find an orthonormal basis for this eigenspace:

Let $\vec{u}_2 = \vec{v}_2$, $\vec{q}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$.

$\vec{u}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_2)\vec{q}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$, $\vec{q}_3 = \frac{\vec{u}_3}{\|\vec{u}_3\|} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$.

Thus, the orthogonal matrix C and the diagonal matrix D are:

$C = [\vec{q}_1, \vec{q}_2, \vec{q}_3] = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) To find the projection matrix P onto W (the eigenspace for $\lambda = 3$), we can use the basis or the orthonormal basis vectors of W.

Let $B = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix}$ then $P = B(B^TB)^{-1}B^T = QQ^T = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

2. Prove or disprove that if A and B are orthogonal $n \times n$ matrices, then AB is orthogonal.

Solution：

It is true! 6-3 example 19(f).