

1. 請框出答案. 2. 不可使用手機、計算器，禁止作弊!

1. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation defined by $T([x, y, z]) = [4x - 2y, -x + 5y + 4z, 5z]^T$.

(a) Find an ordered basis B for \mathbb{R}^3 such that the matrix representation of T relative to B is a diagonal matrix.

(b) Compute $T^{20}([1, 1, 1]^T)$.

Answer:

(a) $B = \underline{\{[2, 1, 0]^T, [1, -1, 0]^T, [4, -2, 1]^T\}}$

(b) $T^{20}([1, 1, 1]^T) = \underline{[-3 \cdot 6^{20} + 4 \cdot 5^{20}, 3 \cdot 6^{20} - 2 \cdot 5^{20}, 5^{20}]^T}$

Solution :

(a) The standard matrix representation of T is

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -1 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

The characteristic equation is:

$$\det(A - \lambda I) = (5 - \lambda)((4 - \lambda)(5 - \lambda) - 2) = (5 - \lambda)(\lambda^2 - 9\lambda + 18) = -(\lambda - 3)(\lambda - 6)(\lambda - 5) = 0$$

The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 6$, and $\lambda_3 = 5$.

For $\lambda_1 = 3$:

$$A - 3I = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvector is $\mathbf{v}_1 = [2, 1, 0]^T$.

For $\lambda_2 = 6$:

$$A - 6I = \begin{bmatrix} -2 & -2 & 0 \\ -1 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvector is $\mathbf{v}_2 = [1, -1, 0]^T$.

For $\lambda_3 = 5$:

$$A - 5I = \begin{bmatrix} -1 & -2 & 0 \\ -1 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvector is $\mathbf{v}_3 = [4, -2, 1]^T$.

Therefore, the ordered basis is $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) To compute $T^{20}([1, 1, 1]^T)$, we first express $\mathbf{v} = [1, 1, 1]^T$ as a linear combination of the basis vectors in B :

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

From the third component, $c_3(1) = 1 \implies c_3 = 1$. Substituting $c_3 = 1$ into the first two components gives:

$$\begin{cases} 2c_1 + c_2 + 4 = 1 \\ c_1 - c_2 - 2 = 1 \end{cases} \implies \begin{cases} 2c_1 + c_2 = -3 \\ c_1 - c_2 = 3 \end{cases}$$

Adding the two equations yields $3c_1 = 0 \implies c_1 = 0$. Then $c_2 = -3$. Thus, $\mathbf{v} = 0\mathbf{v}_1 - 3\mathbf{v}_2 + 1\mathbf{v}_3$.

Now apply T^{20} (which corresponds to multiplying by A^{20}):

$$\begin{aligned} T^{20}(\mathbf{v}) &= A^{20}(-3\mathbf{v}_2 + \mathbf{v}_3) = -3(6^{20})\mathbf{v}_2 + 5^{20}\mathbf{v}_3 \\ &= -3 \cdot 6^{20} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 5^{20} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \cdot 6^{20} + 4 \cdot 5^{20} \\ 3 \cdot 6^{20} - 2 \cdot 5^{20} \\ 5^{20} \end{bmatrix} \end{aligned}$$

2. Prove or disprove that any two $n \times n$ diagonal matrix are similar.

Solution :

Ch 7-2, problem 23 (h).