# 應數一線性代數 2019 秋, 第一次期中考解答

學號: ______________________, 姓名: ________解答________

本次考試共有 8 頁 (包含封面)，有 12 題。如有缺頁或漏題，請立刻告知監考人員。

## 考試須知:

- 請在第一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：誠敬弘遠

誠，一生動念都是誠實端正的。敬，就是對知識的認真尊重。宏，開拓視界，恢宏心胸。遠，任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______________________

以下由閱卷人員填寫

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 10 | 5 | 10 | 10 | 10 | 10 | 10 | 5 | 5 | 10 | 10 | 5 | 100 |
| Score: | | | | | | | | | | | | | |

1. (10 points) Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$$

Compute the following matrices or write DNE if the it is undefined.

(a) $AB$

$$\begin{bmatrix} 4 & 1 \\ 2 & -1 \\ 9 & 2 \end{bmatrix}$$

(b) $BA$.

BA is undefined since the number of columns of B is not equal to the number of rows of A.

(c) $A + 2B$.

$A + 2B$ is undefined since A is $3 \times 3$ and B is $3 \times 2$.

2. (5 points) Determine if the set $W = \{(x, y, z) \in \mathbb{R}^3 | y = x, z = 2x\}$ is a subspace of $\mathbb{R}^3$

Let $w_1 = (x_1, x_1, 2x_1)$ and $w_2 = (x_2, x_2, 2x_2)$ be arbitrary vectors in $W$.
Then for any real number $c$, $cw_1 = (cx_1, cx_1, 2cx_1) \in W$ (by taking $x = cx_1$)
$w_1 + w_2 = (x_1 + x_2, x_1 + x_2, 2x_1 + 2x_2) = (x_1 + x_2, x_1 + x_2, 2x_1 + 2x_2) \in W$ (by taking $x = x_1 + x2$)
So since $W$ is closed under scalar multiplication and vector addition, $W$ is a subspace of $\mathbb{R}^3$

3. (10 points) Consider the following linear system $\begin{cases} x_1 - x_2 + \ \ x_3 + \ \ x_4 = 5 \\ \qquad x_2 - \ \ x_3 + 2x_4 = 8 \\ 2x_1 - x_2 - 3x_3 + 4x_4 = 18 \end{cases}$

(a) Write down the corresponding augmented matrix and reduce it to row-echelon form.

The augmented matrix $[A|\vec{b}] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 5 \\ 0 & 1 & -1 & 2 & 8 \\ 2 & -1 & 3 & 4 & 18 \end{array}\right]$.

After row reducing, we obtain $\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 5 \\ 0 & 1 & -1 & 2 & 8 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$.

(b) Reduce the augmented matrix further to reduced row-echelon form.

After row reducing, we obtain $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 13 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$.

(c) Write down the solution of the original linear system.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 13 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

4. (10 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

(a) Compute the inverse of A and verify that you have the correct inverse.

$$A^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}.$$

And then check $A^{-1}A = I$

(b) Use part (a) to solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ -3 \end{bmatrix}$$

5. (10 points) Is the following set of vectors dependent or independent?

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & -5 & 0 \\ -2 & 3 & 1 \end{bmatrix}$ is row equivalent to $I_3$, so the vectors are independent.

6. (10 points) Find a basis for (a) the nullspace, (b) the column space, and (c) the row space of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix}$$

(a) The reduced row-echelon form of $A$ is $\tilde{A} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

To find a basis for the nullspace of $A$, we must solve $Ax = 0$. Since the third and fifth columns of Ã do not contain pivots,$x_3$ and $x_5$ are free variables. We set $x_3 = r$ and $x_5 = s$.

Then we obtain: $\begin{cases} x_1 = r \quad - \quad s \\ x_2 = -r - 2s \\ x_3 = r \\ x_4 = -r \\ x_5 = s \end{cases}$. Thus a basis for the nullspace of A is the set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) The basis for the column space of A consists of the columns of A corresponding to the columns of Ã with pivots: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(c) A basis for the row space of A consists of the non-zero rows of $\tilde{A}$ : $\{[1, 0, -1, 0, 1], [0, 1, 1, 0, 2], [0, 0, 0, 1, 1]\}$.

7. (10 points) Let$T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T([1, 0, 0]) = [1, 2, 1], T([0, 1, 0]) = [3, 0, 4]$, and $T([1, 0, 1]) = [5, 4, 6]$.

(a) Find the standard matrix representation of $T$.

$T([0, 0, 1]) = T([1, 0, 1]) - T([1, 0, 0]) = [5, 4, 6] - [1, 2, 1] = [4, 2, 5]$.

Thus the standard matrix representation of $T$ is $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 2 \\ 1 & 4 & 5 \end{bmatrix}$

(b) Use the standard matrix representation to find a formula for $T([x_1, x_2, x_3])$.

$$T([x_1, x_2, x_3]) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 + 4x_3 \\ 2x_1 + 2x_3 \\ x_1 + 4x_2 + 5x_3 \end{bmatrix}$$

(c) Find the kernel of $T$.

To find the kernel of $T$, we solve the system $Ax = 0$.

$$\text{The reduced row-echelon form } \tilde{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the third column does not contain a pivot, $x_3$ is a free variable, and we set $x_3 = r$. Then $x_1 = -r$, $x_2 = -r$, and $x_3 = r$, so

$$ker(T) = span\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

(d) Is the linear transformation $T$ invertible? If so, find the standard matrix representation of $T^{-1}$.

$T$ is not invertible since $A$ is not row equivalent to $I_3$

8. (5 points) Suppose that $T$ is a linear transformation with standard matrix representation $A$, and that $A$ is a $7 \times 6$ matrix such that the nullspace of $A$ has dimension 4. What is the dimension of the range of $T$?

Since the nullity of A is equal to 4, the rank of A is equal to 2. Thus the dimension of the range of T is 2.

9. (5 points) If a $7 \times 9$ matrix $A$ has rank 5, find the dimension of the column space of $A$, the dimension of the nullspace of $A$, and the dimension of the row space of $A$.

The dimension of the column space of A is 5, the dimension of the nullspace of A is 4, and the dimension of the row space of A is 5.

10. (10 points) Suppose that the vectors $\vec{v}, \vec{w}$, and $\vec{x}$ are mutually perpendicular (i.e. $\vec{v}$ and $\vec{w}$ are perpendicular, $\vec{v}$ and $\vec{x}$ are perpendicular, and $\vec{w}$ and $\vec{x}$ are perpendicular). Use dot products to find $\|\vec{v}+3\vec{w}+2\vec{x}\|$ in terms of the magnitudes (lengths) of $\vec{v}, \vec{w}$, and $\vec{x}$. Hint: Start by computing $\|\vec{v}+3\vec{w}+2\vec{x}\|^2$.

$\|\vec{v}+3\vec{w}+2\vec{x}\|^2 = (\vec{v}+3\vec{w}+2\vec{x})\cdot(\vec{v}+3\vec{w}+2\vec{x})$
$= \vec{v}\cdot\vec{v}+\vec{v}\cdot 3\vec{w}+\vec{v}\cdot 2\vec{x}+3\vec{w}\cdot\vec{v}+3\vec{w}\cdot 3\vec{w}+3\vec{w}\cdot 2\vec{x}+2\vec{x}\cdot\vec{v}+2\vec{x}\cdot 3\vec{w}+2\vec{x}\cdot 2\vec{x} = \|\vec{v}\|^2+9\|\vec{w}\|^2+4\|\vec{x}\|^2$
Thus $\|\vec{v}+3\vec{w}+2\vec{x}\| = \sqrt{\|\vec{v}\|^2+9\|\vec{w}\|^2+4\|\vec{x}\|^2}$

11. (10 points) In the following transformation, express the standard matrix representation of the given invertible transformation of $\mathbb{R}^2$ into itself as a product of elementary matrices. Use this expression to describe the transformation as a product of one or more reflections, horizontal or vertical expansions or contractions, and shears.

(a) $T(x, y) = [-y, x]$. (Rotation counterclockwise through $\frac{\pi}{2}$)

In column-vector notation, we have $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, which represents a reflection in the line $y = x$ followed by a reflection in the $y$-axis.

(b) $T(x, y) = [-x, -y]$. (Rotation through $\pi$)

In column-vector notation, we have $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, which represents a reflection in the $x$-axis followed by a reflection in the $y$-axis.

12. (5 points) Classify each of the following statements as True or False. No explanation is necessary.

(a) F_ If $A$ is a $2 \times 3$ matrix and $B$ is a $2 \times 4$ matrix, then $AB$ is a $3 \times 4$ matrix.

(b) F_ Any six vectors in $\mathbb{R}^4$ must span $\mathbb{R}^4$.

(c) T_ Every independent subset of $\mathbb{R}^n$ is a subset of some basis for $\mathbb{R}^n$.

(d) T_ If $A$ is a $7 \times 4$ matrix, and if the dimension of the column space of $A$ is 3,then the columns of $A$ are linearly dependent.

(e) T_ If $T$ is a linear transformation, then $T(0) = 0$.