應數一線性代數 2020 春, 第一次期中考

學號: Sol. ________________, 姓名: ________________

本次考試共有 11 頁 (包含封面)，有 10 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 第一部份是必答題，請務必回答每一題。計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 第二部份是選答題，請在其中挑兩題作答。

高師大校訓：誠敬弘遠

誠，一生動念都是誠實端正的。敬，就是對知識的認真尊重。宏，開拓視界，恢宏心胸。遠，任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ________________

第一部份，必答題，請每一題都要做答

1. (10 points) Let

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}$$

Find (if exists) an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$. Also, find the eigenvalues of A^{100}.

(1) The eigenvalue of A^{100} are $(-1)^{100},\ 2^{100},\ 3^{100}$. (2) Is A diagonalizable? Yes

If A diagonalizable, C= $\begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, D= $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and A^{100}= $\begin{bmatrix} 1 & 0 & 0 \\ 1-3^{100} & 2^{100} & 2^{100}-3^{100} \\ -1+3^{100} & 0 & 3^{100} \end{bmatrix}$(不需要化簡).

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ -4 & 2-\lambda & -1 \\ 4 & 0 & 3-\lambda \end{vmatrix} = (-1-\lambda)(2-\lambda)(3-\lambda) \quad \therefore \lambda = -1, 2, 3$$

$\boxed{\lambda=-1}$

$$[A+\lambda I] = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & -1 \\ 4 & 0 & 4 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} & & r \\ & & \downarrow \end{array} \\ \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{cases} 4x+4r=0 \\ 3y+3r=0 \end{cases} \Rightarrow \begin{array}{l} x=-r \\ y=-r \end{array} \quad \therefore \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\boxed{\lambda=2}$

$$A-2I = \begin{bmatrix} -3 & 0 & 0 \\ -4 & 0 & -1 \\ 4 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} & r & \\ & \downarrow & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \therefore x=z=0 \quad \therefore \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda=3$

$$A-3I = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & -1 \\ 4 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} & & r \\ & & \downarrow \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \therefore \begin{array}{l} x=0 \\ y+r=0 \end{array} \quad \therefore \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},\ C^{-1} = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{100} = CD^{100}C^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1-3^{100} & 2^{100} & 2^{100}-3^{100} \\ -1+3^{100} & 0 & 3^{100} \end{bmatrix}$$

2. (15 points) Find the formula for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects in the line $3x + y = 0$.

Answer: $T([x, y]) = \left[\frac{-4x-3y}{5}, \frac{-3x+4y}{5}\right]$

$T([1,-3]) = [1,-3]$ $\quad T([x,y]) = \left(A\begin{bmatrix} x \\ y \end{bmatrix}\right)^T$

$T([3,1]) = (-1)[3,1]$

$\therefore A = CDC^{-1}$, where $C = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $C^{-1} = \frac{1}{10}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$

$\therefore A = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix}$

$A\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4x-3y \\ -3x+4y \end{bmatrix}$

3. (15 points) (a) Solve the system $\begin{cases} x_1' = 2x_1 + 2x_2 \\ x_2' = x_1 + 3x_2 \end{cases}$ $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(b) Find the solution that satisfies the initial condition $x_1(0) = 2, x_2(0) = 5$.

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2 = (\lambda-1)(\lambda-4)$$

$\boxed{\lambda = 1}$

$$A - I = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \therefore \vec{V} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\boxed{\lambda = 4}$

$$A - 4I = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \therefore \vec{V} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\vec{y}' = D\vec{y} \Rightarrow \begin{cases} y_1' = y_1 \\ y_2' = 4y_2 \end{cases} \Rightarrow \begin{cases} y_1 = k_1 e^t \\ y_2 = k_2 e^{4t} \end{cases}$$

$$\vec{x} = C\vec{y} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 e^t \\ k_2 e^{4t} \end{bmatrix} = \begin{bmatrix} -2k_1 e^t + k_2 e^{4t} \\ k_1 e^t + k_2 e^{4t} \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2k_1 e^0 + k_2 e^{4\cdot 0} \\ k_1 e^0 + k_2 e^{4\cdot 0} \end{bmatrix} = \begin{bmatrix} -2k_1 + k_2 \\ k_1 + k_2 \end{bmatrix} \Rightarrow 3 = 3k_1 \Rightarrow \begin{cases} k_1 = 1 \\ k_2 = 4 \end{cases}$$

$$\therefore \vec{x} = \begin{bmatrix} -2e^t + 4e^{4t} \\ e^t + 4e^{4t} \end{bmatrix}$$

4. (15 points) Let the sequence $a_0, a_1, \ldots$ given by $a_0 = 0, a_1 = 1$, and $a_k = a_{k-1} + \frac{3}{4}a_{k-2}$ for $k \geq 2$.
(1) Find the matrix A that can be used to generate this sequence. (2) Estimate(估計) a_k for large k.

Answer: $A=$ $\begin{bmatrix} 1 & 3/4 \\ 1 & 0 \end{bmatrix}$, $a_k=$ $\frac{1}{8}\left[3\left(\frac{3}{2}\right)^k - 3\left(\frac{-1}{2}\right)^k\right]$.

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 3/4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k-1} \\ a_{k-2} \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3/4 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 3/4 = (\lambda - 3/2)(\lambda + \tfrac{1}{2}) \quad \therefore \lambda = \tfrac{3}{2}, \tfrac{-1}{2}$$

$\boxed{\lambda = \frac{3}{2}}$

$$A - \tfrac{3}{2}I = \begin{bmatrix} -\frac{1}{2} & 3/4 \\ 1 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \quad \therefore \vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$\boxed{\lambda = -\frac{1}{2}}$

$$A + \tfrac{1}{2}I = \begin{bmatrix} 3/2 & 3/4 \\ 1 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \quad \therefore \vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3/2 & 0 \\ 0 & -1/2 \end{bmatrix}, \quad C^{-1} = \frac{1}{8}\begin{bmatrix} 2 & +1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/8 \\ -1/4 & 3/8 \end{bmatrix}$$

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = A^k \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} (\frac{3}{2})^k & 0 \\ 0 & (\frac{-1}{2})^k \end{bmatrix} \begin{bmatrix} 1/4 & 1/8 \\ -1/4 & 3/8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(\frac{3}{2})^k & (-1)(\frac{-1}{2})^k \\ * & * \end{bmatrix} \begin{bmatrix} 1/4 & 1/8 \\ -1/4 & 3/8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} * & \frac{3}{8}(\frac{3}{2})^k - \frac{3}{8}(\frac{-1}{2})^k \\ * & * \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{3}{8} \begin{bmatrix} (\frac{3}{2})^k - (\frac{-1}{2})^k \\ * \end{bmatrix} \qquad \therefore a_k = \frac{3}{8}\left[\left(\frac{3}{2}\right)^k - \left(\frac{-1}{2}\right)^k\right]$$

5. (10 points) Find the projection of $\underset{\vec{b}}{[1,\ 0,\ 0]}$ on the subspace $W = \mathrm{sp}(\underset{\vec{v}_1}{[2,\ 1,\ 1]},\ \underset{\vec{v}_2}{[1,\ 0,\ 2]})$ in $\mathbb{R}^3$

Answer: $[5/7\ \ 3/7\ \ 1/7]$

$\boxed{\text{Method 1}}$

let $\vec{u} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = [2, -3, -1]$

$\therefore W^{\perp} = \mathrm{sp}(\vec{u})$, $\vec{b}_{W^\perp} = \dfrac{\vec{b}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \left[\frac{2}{7}, \frac{-3}{7}, \frac{-1}{7}\right]$

$\vec{b}_W = \vec{b} - \vec{b}_{W^\perp} = [1, 0, 0] - \left[\frac{2}{7}, \frac{-3}{7}, \frac{-1}{7}\right] = \left[\frac{5}{7}, \frac{3}{7}, \frac{1}{7}\right]$

$\boxed{\text{Method 2}}$ (follow sec 6.1. example 3)

$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ $\therefore \mathrm{null}(A) = \mathrm{sp}([-2, 3, 1])$

$\left[\begin{array}{ccc|c} \vec{v}_1^{T} & \vec{v}_2^{T} & \vec{u}^{T} & \vec{b}^{T} \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/7 \\ 0 & 1 & 0 & -1/7 \\ 0 & 0 & 1 & -1/7 \end{array}\right]$

$\therefore \vec{b} = \underbrace{\left[\frac{3}{7}\vec{v}_1 + \left(\frac{-1}{7}\right)\vec{v}_2\right]}_{\vec{b}_W} + \underbrace{\left(\frac{-1}{7}\right)\vec{u}}_{\vec{b}_{W^\perp}}$

$\therefore \vec{b}_W = \frac{3}{7}\vec{v}_1 + \left(\frac{-1}{7}\right)\vec{v}_2 = [5/7\ \ 3/7\ \ 1/7]$

6. (15 points) Use Gram-Schmidt process to find an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by [1, 0, 1, 0], [1, 1, 1, 0], [1, 0, 1, 1]. Find the QR-factorization of A, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer:
Q= $\begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, R= $\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, an orthonormal basis = $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

① $\vec{v}_1 = \vec{a}_1$, $\vec{q}_1 = \dfrac{\vec{v}_1}{|\vec{v}_1|} = \dfrac{1}{\sqrt{2}}\vec{v}_1$, let $W_1 = sp(\vec{v}_1)$ $\Rightarrow \begin{cases} \vec{v}_1 = \sqrt{2}\,\vec{q}_1 \\ \vec{a}_1 = \sqrt{2}\,\vec{q}_1 \end{cases}$ $\vec{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

② $\vec{v}_2 = \vec{a}_2 - \vec{a}_{2_{W_1}} = \vec{a}_2 - \dfrac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \vec{a}_2 - \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\Rightarrow \begin{cases} \vec{v}_2 = \vec{q}_2 \\ \vec{v}_2 = \vec{a}_2 - \vec{v}_1 \end{cases}$ $\vec{q}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{q}_2 = \vec{v}_2 / |\vec{v}_2| = \vec{v}_2$, let $W_2 = sp(\vec{v}_2)$ $\therefore \vec{a}_2 = \vec{v}_1 + \vec{v}_2 = \sqrt{2}\,\vec{q}_1 + \vec{q}_2$

③ $\vec{v}_3 = \vec{a}_3 - \vec{a}_{3_{W_2}} = \vec{a}_3 - \dfrac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \dfrac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \vec{a}_3 - \vec{v}_1 - 0 \cdot \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{q}_3 = \vec{v}_3 / |\vec{v}_3| = \vec{v}_3$ $\Rightarrow \begin{cases} \vec{q}_3 = \vec{v}_3 \\ \vec{v}_3 = \vec{a}_3 - \vec{v}_1 \end{cases}$ $\Rightarrow \vec{a}_3 = \vec{v}_1 + \vec{v}_3 = \sqrt{2}\,\vec{q}_1 + \vec{q}_3$

$\therefore Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} = QR = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

第二部份，選答題，請勾選兩題 予以評分

☐ 本題我要作答

7. (10 points) Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Prove that there is one and only one vector $\vec{p}$ in W such that $\vec{b}-\vec{p}$ is perpendicular(垂直) to every vector in W.

Assume $\exists\ \vec{P_1}, \vec{P_2} \in W$ s.t. $\vec{b}-\vec{P_1},\ \vec{b}-\vec{P_2} \in W^{\perp}$

$\forall\ \vec{V} \in W$

$0 = \vec{V}\cdot(\vec{b}-\vec{P_1}) = \vec{V}\cdot\vec{b} - \vec{V}\cdot\vec{P_1} \Rightarrow \vec{V}\cdot\vec{b} = \vec{V}\cdot\vec{P_1}$

$0 = \vec{V}\cdot(\vec{b}-\vec{P_2}) = \vec{V}\cdot\vec{b} - \vec{V}\cdot\vec{P_2} \Rightarrow \vec{V}\cdot\vec{b} = \vec{V}\cdot\vec{P_2}$

$\therefore\ 0 = \vec{V}\cdot\vec{b} - \vec{V}\cdot\vec{b} = \vec{V}\cdot\vec{P_1} - \vec{V}\cdot\vec{P_2} = \vec{V}\cdot(\vec{P_1}-\vec{P_2})$

$\because\ \forall\ \vec{V}\in W \Rightarrow 0 = \vec{V}\cdot(\vec{P_1}-\vec{P_2})\ \therefore\ \vec{P_1}-\vec{P_2} \in W^{\perp}$

$\because$ W is a vector space and $\vec{P_1}, \vec{P_2} \in W$ $\therefore\ \vec{P_1}-\vec{P_2} \in W$

$\therefore\ \vec{P_1}-\vec{P_2} \in W \cap W^{\perp} = \{\vec{0}\}$

$\therefore\ \vec{P_1} = \vec{P_2}$

☐ 本題我要作答

8. (10 points) The **trace** of an $n \times n$ matrix A is defined by

$$tr(A) = a_{11} + a_{22} + ... + a_{nn}.$$

Let the characteristic polynomial $p(A)$ factor(因式分解) into linear factors(一次因式), so that A has n (not necessarily(必須) distinct(不同)) eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Prove that

$$\begin{aligned} tr(A) &= (-1)^{n-1}(\text{Coefficient(係數) of } \lambda^{n-1} \text{ in } p(\lambda)) \\ &= \lambda_1 + \lambda_2 + ... + \lambda_n \end{aligned}$$

① $P(\lambda) = |A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$

$\therefore$ the coef. of λ^{n-1} in $P(\lambda)$ = the coef of λ^{n-1} in $(\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$

$= (-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

② let $B = A - \lambda I$.

$\det(B) = b_{11}|B_{11}| - b_{12}|B_{12}| + b_{13}|B_{13}| - \cdots + (-1)^{n+1} b_{1n}|B_{1n}|$

Note that for $j \neq 1$, B_{1j} has only $n-2$ elements contains λ

~~i.e. $|B_{1j}|$ is at most degree n-2~~

i.e. the degree of $|B_{1j}|$ is at most $(n-2)$

$\therefore$ the coef. of λ^{n-1} in $P(\lambda)$ $(= \det(B))$

= the coef. of λ^{n-1} in $b_{11}|B_{11}| = (a_{11} - \lambda)\begin{vmatrix} a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33}-\lambda & & \vdots \\ \vdots & & \ddots & \\ a_{n2} & \cdots & & a_{nn}-\lambda \end{vmatrix}$

(by the same reason)

= the coef. of λ^{n-1} in $(a_{11} - \lambda)(a_{22} - \lambda)\begin{vmatrix} a_{33}-\lambda & a_{34} & \cdots & a_{3n} \\ a_{43} & a_{44}-\lambda & \cdots & a_{4n} \\ & & & \vdots \\ a_{n3} & \cdots & & a_{nn}-\lambda \end{vmatrix}$

= the coef. of λ^{n-1} in $(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)\begin{vmatrix} a_{44}-\lambda & & a_{ij} \\ a_{ij} & \ddots & a_{nn}-\lambda \end{vmatrix}$

=
$\vdots$

= the coef of λ^{n-1} in $(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$

$= (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}) = (-1)^{n-1} tr(A)$

☐ 本題我要作答

9. (10 points) Prove that, for every square(正方形) matrix A all of whose eigenvalues are real, the product of its eigenvalues is $\det(A)$

let $P(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$, where $\{\lambda_i\}$ are eigenvalues of A

$P(0) = \det(A - 0I) = \det(A)$

$\parallel$

$(\lambda_1 - 0)(\lambda_2 - 0) \cdots (\lambda_n - 0) = \lambda_1 \lambda_2 \cdots \lambda_n$

$\therefore \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

☐ 本題我要作答

10. (10 points) Prove that, if a matrix is diagonalizable(可對角線化), so is its transpose(轉置).

A: diagonalizable $\Rightarrow \exists C$: invertible and $\exists D$: diagonal

s.t. $A = CDC^{-1}$

$A^T = (CDC^{-1})^T = (C^{-1})^T D^T C^T$, Note D^T also a diagonal matrix.

Claim: $(C^{-1})^T = (C^T)^{-1}$

pf. $I = I^T = (CC^{-1})^T = (C^{-1})^T C^T$

$\because$ the uniquness of inverse matrix

$\therefore (C^{-1})^T = (C^T)^{-1}$

$\therefore A^T = (C^T)^{-1} D^T C^T$ is diagonalizable.

學號: ____________, 姓名: ____________, 以下由閱卷人員填寫

Question:	1	2	3	4	5	6	Total
Points:	10	15	15	15	10	15	80
Score:							

7	8	9	10	Total
10	10	10	10	20