應數一線性代數 **2020** 秋, 第一次期中考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 8 頁 (包含封面)，有 13 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：誠敬弘遠

誠，一生動念都是誠實端正的。敬，就是對知識的認真尊重。宏，開拓視界，恢宏心胸。遠，任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______________________________

1. (5 points) Find the value of x such that $[x, -3, 5]$ is perpendicular to $[-1, 3, 4]$

   Answer: x= 11

   $[-1, 3, 4] \cdot [x, -3, 5] = -x - 9 + 40 = -x + 11 = 0$, $x = 11$

2. (10 points) Solve the given linear system and express the solution set.

$$\begin{cases} x_1 - x_2 + \ x_3 + \ x_4 = 5 \\ \qquad x_2 - \ x_3 + 2x_4 = 8 \\ x_1 \qquad - 4x_3 + 3x_4 = 13 \end{cases}$$

   Answer: the solution set is $\left\{ \begin{bmatrix} 13 \\ 8 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 5 \\ 0 & 1 & -1 & 2 & 8 \\ 1 & 0 & -4 & 3 & 13 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 13 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

   Let $x_4 = r$, we get $x_3 = 0, x_2 + 2r = 8, x_1 + 3r = 13$. Thus $x_1 = 13 - 3r, x_2 = 8 - 2r, x_3 = 0, x_4 = r$. Then solution are

$$\left\{ \begin{bmatrix} 13 - 3r \\ 8 - 2r \\ 0 \\ r \end{bmatrix} \middle| r \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 13 \\ 8 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$$

3. (10 points) Assume the the matrix A can be row reduces to H, please answer the following questions.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) the **rank** of matrix A, is ___4___.

(b) a basis for the **row space** of A is ___[1, 0, 0, 0,-3], [0,1,0,0,3],[0,0,1,0,-1], [0, 0, 0, 1, 1]___.

(c) a basis for the **column space** of A is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$.

(d) a basis for the **nullspace** of A is $\begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

Let $x_5 = r$, we get $x_4 + r = 0, x_3 - r = 0, x_2 + 3r = 0, x_1 - 3r = 0$. Thus $x_1 = 3r, x_2 = -3r, x_3 = r, x_4 = -r, x_5 = r$. Then homogeneous solution are

$$\left\{ \begin{bmatrix} 3r \\ -3r \\ r \\ -r \\ r \end{bmatrix} \middle| r \in \mathbb{R} \right\} = \left\{ r \begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$$

4. (5 points) If a 8×11 matrix A has rank 5, find the dimension of the column space of A, the dimension of the nullspace of A, and the dimension of the row space of A.

[the dimension of the column space of A] = [the dimension of the row space of A] = [the rank of A] =5.

[the dimension of the nullspace of A] = [the number of columns in A] - [the rank of A] =11-5=6

5. (10 points) Given set $S = \{[-2,2,3,0],[1,-2,1,0],[-1,0,4,0]\}$ in $\mathbb{R}^4$.

   (a) Determine whether the set S is linearly dependent or linearly independent. If it is linearly dependent, find a basis for $sp(S)$.

   Answer: $\{[-2,2,3,0],[1,-2,1,0]\}$.

   (b) Enlarge the basis you found in part (a) to be a basis for $\mathbb{R}^4$.

$$\begin{bmatrix} -2 & 1 & -1 & 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1/8 & 1/4 & 0 \\ 0 & 1 & 1 & 0 & -3/8 & 1/4 & 0 \\ 0 & 0 & 0 & 1 & 5/8 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A basis for $\mathbb{R}^4$ is $\{[-2,2,3,0],[1,-2,1,0],[1,0,0,0],[0,0,0,1]\}$

6. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T([1,0,0]) = [1,2,1], T([0,1,0]) = [3,0,4]$, and $T([1,0,1]) = [5,4,6]$.

   (a) Find the standard matrix representation of T.

   $T([0,0,1]) = T([1,0,1]) - T([1,0,0]) = [5,4,6] - [1,2,1] = [4,2,5]$.

   Thus the standard matrix representation of T is $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 2 \\ 1 & 4 & 5 \end{bmatrix}$

   (b) Use the standard matrix representation to find a formula for $T([x_1, x_2, x_3])$.

   $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 + 4x_3 \\ 2x_1 + 2x_3 \\ x_1 + 4x_2 + 5x_3 \end{bmatrix}$, $T([x_1, x_2, x_3]) = [x_1 + 3x_2 + 4x_3, 2x_1 + 2x_3, x_1 + 4x_2 + 5x_3]$

   (c) Find the kernel of T.

   To find the kernel of T, we solve the system $Ax = 0$.

   $$\text{The reduced row-echelon form } \tilde{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Since the third column does not contain a pivot,x_3 is a free variable, and we set $x_3 = r$. Then $x_1 = -r, x_2 = -r$, and $x_3 = r$, so

   $$ker(T) = span\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

   $\begin{cases} x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$, hence, the kernel of T is $\left\{ \begin{bmatrix} -2r \\ -r \\ r \end{bmatrix} \middle| r \in \mathbb{R} \right\}$

   (d) Is the linear transformation T invertible? If so, find the standard matrix representation of T^{-1}.

   Since the standard matrix representation of T is not invertible (not row equivalent to I_3), T is not invertible as well.

7. (10 points) (a) Compute the inverse of A and verify that you have the correct inverse.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix} \qquad \text{Answer: } A^{-1} = \begin{bmatrix} \frac{-1}{3} & \frac{-1}{3} & \frac{2}{3} \\ 1 & 1 & -1 \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix}.$$

(b) Use part (a) to solve

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & \frac{-1}{3} & \frac{2}{3} \\ 1 & 1 & -1 \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} \\ 2 \\ \frac{-1}{3} \end{bmatrix}$$

8. (5 points) Determine if the set $W = \{(x, y, z) \in \mathbb{R}^3 | z = 3x + 2\}$ is a subspace of $\mathbb{R}^3$.

Let $\vec{u} = [x, y, 3x + 2], \vec{v} = [a, b, 3a + 2]$.
$\vec{u} + \vec{v} = [x, y, 3x + 2] + [a, b, 3a + 2] = [x + a, y + b, 3(x + a) + 4]$ which is clearly not in W.
Hence W is not closed under vector addition. W is not a subpace of $\mathbb{R}^3$

9. (5 points) Circle True or False. Read each statement in original Greek before answering.
   (a) True **False** If A is a 2×3 matrix and B is a 2×4 matrix, then AB is a 3×4 matrix.
   (b) True **False** Any six vectors in $\mathbb{R}^4$ must span $\mathbb{R}^4$.
   (c) **True** False If T is a linear transformation, then $T(0) = 0$.
   (d) True **False** No vector is its own additive inverse.
   (e) True **False** If $\{v_1, v_2, ..., v_n\}$ generates V, then each $v \in V$ is a unique linear combination of vectors in this set.

10. (5 points) Let F bet he set of all real-valued functions on a (nonempty) set S; that is , let F be the set of all functions mapping S into $\mathbb{R}$. For $f, g \in F$, let the sum $f \oplus g$ of two functions f and g in F, and for any scalar r, let scalar multiplication be defined below. Is this set a vector space?

$$(f \oplus g)(x) = \max\{f(x), g(x)\} \quad \text{for all } x \in S$$
$$(rf)(x) = rf(x) \quad \text{for all } x \in S$$

Define $(f \oplus g) = \max\{f(x), g(x)\}$, for all $x \in \mathbb{R}$ and $(rf)(x) = rf(x)$, for all $x \in \mathbb{R}$.
Assume $z(x)$ is the $\vec{0}$, that is for all $f(x)$, $z(x) = f(x) \oplus (-f)(x) = \max\{f(x), (-f)(x)\} = \max\{f(x), -f(x)\}$.
Let $f(x) = 1, z(x) = f(x) \oplus (-f)(x) = \max\{1, -1\} = 1$.
However, by **A3**, $z(x) \oplus (-f)(x) = (-f)(x) = -1 \neq \max\{1, -1\}$.
Therefore, $\vec{0}$ does not exists.

11. (5 points) Determine the following set is a subspace of the given vector space. The set of all functions f such that $f(0) = 1$ in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

Let f, g are two functions satisfies the assumption.
$(f + g)(0) = f(0) + g(0) = 1 + 1 = 2 \neq 1$. Hence that is not a vector space.

12. (10 points) Let W_1 and W_2 be two subspace of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace.

Clearly $W_1 \cap W_2$ is nonempty; it contains 0.
Let $\vec{v}, \vec{w} \in (W_1 \cap W_2)$. Then $\vec{v}, \vec{w} \in W_1$ and $\vec{v}, \vec{w} \in W_2$, so $\vec{v} + \vec{w} \in W_1$ and $\vec{v} + \vec{w} \in W_2$ since W_1 and W_2 are subspaces.
Thus $\vec{v}+\vec{w} \in (W_1 \cap W_2)$. Similarly, $r\vec{v} \in W_1$ and $r\vec{v} \in W_2$. Since W_1 and W_2 are subspaces. Thus $r\vec{v} \in (W_1 \cap W_2)$.
Thus W_1 and W_2 are subspaces. Thus $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$

13. (10 points) Prove that the given relation holds for all vectors, matrices and scalars for which the expression are defined.

$$(AB)^T = B^T A^T$$

The $(i,j)^{th}$ entry of $(AB)^T$ is the $(j,i)^{th}$ entry in AB, which is

$$\begin{aligned} & (j^{th} \text{ row of } A) \cdot (i^{th} \text{ column of } B) \\ = & (i^{th} \text{ column of } B) \cdot (j^{th} \text{ row of } A) \\ = & (i^{th} \text{ row of } B^T) \cdot (j^{th} \text{ column of } A^T) \end{aligned}$$

which is the $(i,j)^{th}$ entry of $B^T A^T$. Since $(AB)^T$ and $B^T A^T$ have the same size, they are equal.

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Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
Points:	5	10	10	5	10	10	10	5	5	5	5	10	10	100
Score:														