應數一線性代數 2021 春, 期中考試卷 A 解答

學號: ________________________________, 姓名: ________________________________

本次考試共有 9 頁 (包含封面)，有 9 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -7 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix}$$

Find (if exists) an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$. Also, find the eigenvalues of A^{100}.

(1) Is A diagonalizable? ___No!___. If A diagonalizable, C= ___________, D=___________.

(2) The eigenvalue of A are ___-2, 2, 2___. The eigenvalue of A^{100} are ___$2^{1}00$___.

Answer:

$|A - \lambda I| = (2-\lambda)(\lambda^2 - 4)$, $\lambda = -2, 2, 2$

$$A - 2I = \begin{bmatrix} -3 & 0 & 1 \\ -7 & 0 & 5 \\ 3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -3 & 0 & 1 \\ 0 & 0 & 8/3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix}, \text{ for } s \neq 0$$

Hence the algebra multiplicity of 2 is 2, but the geometry multiplicity of 2 is 1. Since they are NOT EQUAL, A is not diagonalizable.

2. (15 points) Find the formula for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects in the line $x + 5y = 0$.

Answer: $T([x, y]) =$ $\frac{1}{13}[12x - 5y, -5x - 12y]$

Answer:

By the definition of " reflects in the line $x + 5y = 0$ ", we know T maps the vector $[5, -1]$ onto $[5, -1]$, and maps the vector $[1, 5]$ onto $[-1, -5]$. Thus,

$$T([5, -1]) = [5, -1], T([1, 5]) = [-1, -5],$$

Let the s.m.r. of T is A, we got the eigenvalues of A are 1, -1, and the corresponding eigenvectors are $\begin{bmatrix} 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$$A = \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}^{-1} = \frac{1}{13} \begin{bmatrix} 12 & -5 \\ -5 & -12 \end{bmatrix}$$

$$A * \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 12x - 5y \\ -5x - 12y \end{bmatrix}$$

3. (15 points) (a) Solve the system $\begin{cases} x_1' = 3x_1 + 2x_2 \\ x_2' = x_1 + 2x_2 \end{cases}$

(b) Find the solution that satisfies the initial condition $x_1(0) = 2, x_2(0) = 5$.

Answer: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{14}{3}e^{4t} - \frac{8}{3}e^{t} \\ \frac{7}{3}e^{4t} + \frac{8}{3}e^{t} \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

Then the eigenvalues of A are 4, 1 and the corresponding eigenvectors are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} y_1' = 4y_1 \\ y_2' = y_2 \end{cases} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k_1 e^{4t} \\ k_2 e^{t} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 e^{4t} \\ k_2 e^{t} \end{bmatrix} = \begin{bmatrix} 2k_1 e^{4t} - k_2 e^{t} \\ k_1 e^{4t} + k_2 e^{t} \end{bmatrix}$$

Since

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2k_1 e^{0} - k_2 e^{0} \\ k_1 e^{0} + k_2 e^{0} \end{bmatrix} \Rightarrow \begin{cases} k_1 = \frac{7}{3} \\ k_2 = \frac{8}{3} \end{cases}$$

Hence

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{14}{3}e^{4t} - \frac{8}{3}e^{t} \\ \frac{7}{3}e^{4t} + \frac{8}{3}e^{t} \end{bmatrix}$$

4. (10 points) Find the projection matrix P for the plane $W : 2x + 2y + z = 0$ and then find the projection of $\vec{b} = [4, 2, -1]$ on the plane.

Answer: $\vec{b}_W =$ ____________________, $P =$ ____________________.

The basis for W are $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{bmatrix}$, we have

$$(A^T A)^{-1} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{9} \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$$P = A(A^T A)^{-1} A^T = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$

$$\vec{b}_W = P\vec{b} == \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 14 \\ -4 \\ -20 \end{bmatrix}$$

5. (10 points) Find the least-square solution of the below system.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}$$

Answer: The least-square solution = ________________.

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

The given system $A\vec{x} = \vec{b}$, we can rewrite is as $A^T A\vec{x} = A^T\vec{b}$.

$$A^T A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \ A^T\vec{b} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix},$$

Solve

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

We have the least-square solution is

$$\frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

6. (15 points) Use Gram-Schmidt process to find an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by [1, 1, 0, 0], [1, 1, -1, 0], [1, 0, 1, 1] and then use it to find the QR-factorization of A, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer : Q=__________, R=__________, an orthonormal basis = $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$

By Gram-Schmidt process. Let $\vec{a}_1 = [1,1,0,0], \vec{a}_2 = [1,1,-1,0], \vec{a}_3 = [1,0,1,1]$

$$\vec{v}_1 = \vec{a}_1 = [1,1,0,0],\ \vec{q}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0] \tag{1}$$

$$\vec{v}_2 = \vec{a}_2 - (\vec{a}_2 \cdot \vec{q}_1)\vec{q}_1 = \vec{a}_2 - \sqrt{2}\vec{q}_1 = [0,0,-1,0] \tag{2}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{|\vec{v}_2|} = [0,0,-1,0] \tag{3}$$

$$\vec{v}_3 = \vec{a}_3 - (\vec{a}_3 \cdot \vec{q}_1)\vec{q}_1 - (\vec{a}_3 \cdot \vec{q}_2)\vec{q}_2 = \vec{a}_3 - \frac{\sqrt{2}}{2}\vec{q}_1 + \vec{q}_2 = [\frac{1}{2}, \frac{-1}{2}, 0, 1] \tag{4}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{|\vec{v}_3|} = [\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}] \tag{5}$$

Then by (1), we get (6). By (2) and (3), we get (7). By (4) and (5), we get (8).

$$\vec{a}_1 = \sqrt{2}\,\vec{q}_1 \Rightarrow \begin{bmatrix} \vec{a}_1^T \end{bmatrix} = \begin{bmatrix} \vec{q}_1^T \end{bmatrix} \begin{bmatrix} \sqrt{2} \end{bmatrix} \tag{6}$$

$$\vec{a}_2 = \sqrt{2}\,\vec{q}_1 + \vec{v}_2 = \sqrt{2}\,\vec{q}_1 + \vec{q}_2 \Rightarrow \begin{bmatrix} \vec{a}_2^T \end{bmatrix} = \begin{bmatrix} \vec{q}_1^T & \vec{q}_2^T \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix} \tag{7}$$

$$\vec{a}_3 = \frac{\sqrt{2}}{2}\vec{q}_1 - \vec{q}_2 + \vec{v}_3 = \frac{\sqrt{2}}{2}\vec{q}_1 - \vec{q}_2 + \frac{\sqrt{3}}{\sqrt{2}}\vec{q}_3 \Rightarrow \begin{bmatrix} \vec{a}_3^T \end{bmatrix} = \begin{bmatrix} \vec{q}_1^T & \vec{q}_2^T & \vec{q}_3^T \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -1 \\ \frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix} \tag{8}$$

Therefore,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = QR = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{6} \\ 0 & -1 & 0 \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix}$$

7. (10 points) Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Prove that there is one and only one vector $\vec{p}$ in W such that $\vec{b} - \vec{p}$ is perpendicular(垂直) to every vector in W.

Assume there're two vectors $\vec{p}_1, \vec{p}_2 \in W$ such that $\vec{b} - \vec{p}_1$ and $\vec{b} - \vec{p}_2$ are both perpendicular to every vector in W . i.e. $\vec{b} - \vec{p}_1$ and $\vec{b} - \vec{p}_2$ are both in $W^{\perp}$.

For all vector $\vec{v} \in W$

$$0 = \vec{v} \cdot (\vec{b} - \vec{p}_1) = \vec{v} \cdot \vec{b} - \vec{v} \cdot \vec{p}_1 \therefore \vec{v} \cdot \vec{b} = \vec{v} \cdot \vec{p}_1$$

$$0 = \vec{v} \cdot (\vec{b} - \vec{p}_2) = \vec{v} \cdot \vec{b} - \vec{v} \cdot \vec{p}_2 \therefore \vec{v} \cdot \vec{b} = \vec{v} \cdot \vec{p}_2$$

$$\therefore \vec{v} \cdot (\vec{p}_1 - \vec{p}_2) = 0$$

$$\therefore \vec{p}_1 - \vec{p}_2 \in W^{\perp}$$

Note that W is a vector space and $\vec{p}_1, \vec{p}_2 \in W$, we will have $\vec{p}_1 - \vec{p}_2 \in W^{\perp}$. Since $\vec{p}_1 - \vec{p}_2$ in both W and $W^{\perp}$, we can easily checked that $\vec{p}_1 - \vec{p}_2 = \vec{0}$.

8. (10 points) Let A and C be orthogonal $n \times n$ matrices. Show that CAC^{-1} is orthogonal.

A and C be orthogonal matrices, i.e. $A^TA = C^TC = I$. We also know that $C^{-1} = C^T$, i.e. $CAC^{-1} = CAC^T$

$(CAC^T)^T(CAC^T) = (C^T)^TA^TC^TCAC^T = CA^TC^TCAC^T = CA^T(C^TC)AC^T$

$= CA^TIAC^T = C(A^TA)C^T = CIC^T = (C^TC)^T = I^T = I.$

We have $CAC^T = CAC^{-1}$ is orthogonal.

9. (5 points) Circle True or False. Read each statement in original Greek before answering.

(a) **[True]** False If $\vec{v}$ is an eigenvector of an invertible matrix A, then $c\vec{v}$ is an eigenvector of A^{-1} for all nonzero scalar c.

Since $A\vec{v} = \lambda\vec{v}$, we have $A(c\vec{v}) = c(A\vec{v}) = c(\lambda\vec{v}) = \lambda(c\vec{v})$ and $c\vec{v} \neq \vec{0}$ when $\vec{v} \neq \vec{0}$ and c is a nonzero scalar.

(b) **[True]** False Every $n \times n$ real symmetric matrix is real diagonalizable.

By Theorem 6.8 from the textbook.

(c) True **[False]** The intersection of W and $W^\perp$ is empty.

$\vec{0} \in W \cap W^\perp$ since $\vec{0}$ belong to every vector space.

(d) True **[False]** A square matrix is orthogonal if its column vectors are orthogonal.

A square matrix is orthogonal if its column vectors are "orthonormal".

For example $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are orthogonal. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. $A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(e) True **[False]** The least-square solution vector of $A\vec{x} = \vec{b}$ is the projection of $\vec{b}$ on the column space of A.

The least-square solution vector of $A\vec{x} = \vec{b}$ is the vector $\vec{y}$ such that $A\vec{y}$ is the projection of $\vec{b}$ on the column space of A.

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	15	15	10	10	15	10	10	5	100
Score:										